Lecture II.

Recall:

Ex. $(1+t^2)y' + 4ty = (1+t^2)^{-2}$, $y(1)=0$.

Step 1: Rewrite it as

$$y' + \frac{4t}{1+t^2}y = (1+t^2)^{-3}$$

then $a(t) = -\frac{4t}{1+t^2}$, $f(t) = (1+t^2)^{-3}$.

Step 2: Multiply by the integration factor

$$u(t) = e^{-\int a(t)\,dt} = e^{-\int -\frac{4t}{1+t^2}\,dt}$$

$$= e^{\int \frac{4t}{1+t^2}\,dt} = e^{2\ln(1+t^2)} = (1+t^2)^2.$$

(PS: If use $s=t^2$, $ds = 2t\,dt$.

$$\int \frac{4t}{1+t^2}\,dt = \int \frac{2(2t\,dt)}{1+t^2} = \int \frac{2\,ds}{1+s}$$

$$= 2\ln(1+s) = 2\ln(1+t^2).$$)

we get $\left((1+t^2)^2 y(t)\right)' = (1+t^2)^2(1+t^2)^{-3} = (1+t^2)^{-1}$

Step 3: Integrate the equation to obtain

$$(1+t^2)^2 y(t) = \int (1+t^2)^{-1}\,dt = \arctan t + C$$

Step 4: Solve $y(t) = (1+t^2)^{-2}\arctan t + C(1+t^2)^{-2}$.

For C, plug in $y(1)=0$. then $0 = (1+1)^{-2}\arctan 1 + C(1+1)^{-2}$

$$= \frac{\pi}{8} + \frac{C}{4} \Rightarrow C = -\frac{\pi}{2}. \quad \square$$

Structure Of the Solution

For $y' = a(t)y + f(t)$. $(*)$

and y_p is one solution of it.

Consider the homogeneous equation

$$y' = a(t)y \qquad (\Delta)$$

and y_h is one solution of it.

then $y_p + Ay_h$ is solution of $(*)$ for every A.

Why? Let's check. plug in $y_p + Ay_h$.

Left side of $(*)$ $= (y_p + Ay_h)' = y_p' + Ay_h'$

Since y_p is solution of $(*)$, $y_p' = a(t)y_p + f(t)$

y_h is solution of (Δ), $y_h' = a(t)y_h$.

so $= a(t)y_p + f(t) + Aa(t)y_h$

$= a(t)(y_p + Ay_h) + f(t)$.

$=$ Right side of $(*)$. $\square$

Summarize this 2 fact into a theorem.

Thm. Suppose that y_p is a particular solution to the inhomogeneous equation $y' = a(t)\,y + f(t)$,

and that y_h is a particular solution to the associated homogeneous equation.

Then every solution to the inhomogeneous equation is of the form

$$y(t) = y_p(t) + A\,y_h(t),$$

where A is an arbitrary constant.

Indeed, every solution to $(*)$ will be

$y_p + Ay_h$, for some A.

Why? Let's check.

If y_q is another solution to $(*)$.

$$\text{so} \quad y_p' = a(t)\, y_p + f(t) \qquad (1)$$

$$y_q' = a(t)\, y_q + f(t) \qquad (2)$$

Use (2) − (1),

get $(y_q - y_p)' = a(t)(y_q - y_p)$.

hence $y_q - y_p$ is solution of (Δ).

so $y_q - y_p = A y_h$. for some A.

then $y_q = y_p + A y_h$.

Differential Forms and Differential Equations

We will consider more general equations

$$P(x,y) + Q(x,y)\,y' = 0. \qquad (\triangle)$$

Rewrite it as

$$P(x,y) + Q(x,y)\,\frac{dy}{dx} = 0.$$

$$\Rightarrow \quad P(x,y)\,dx + Q(x,y)\,dy = 0. \quad (*)$$

Defn. A differential form in two variables x and y is an expression of the type

$$\omega = P(x,y)\,dx + Q(x,y)\,dy.$$

Ex. $2\,dx$, $\quad 2\,dx + y\,dy$, $\quad x\,dx$.

So $\boxed{y(x) \text{ is solution to } P(x,y) + Q(x,y)\,y' = 0} \iff \boxed{\text{Differential form } P(x,y)\,dx + Q(x,y)\,dy = 0}$

For this reason, we will consider $(*)$ as another way of writing the differential equation in $(\triangle)$.

Ex. Consider the differential equation

$$\omega = x\,dx + y\,dy = 0, \quad \text{or} \quad \frac{dy}{dx} = -\frac{x}{y}.$$

Show that this equation has solutions defined implicitly by the equation $x^2+y^2 = C$.

Check: $\quad y = \pm\sqrt{C-x^2}$.

Just check $y = \sqrt{C-x^2} = (C-x^2)^{\frac{1}{2}}$

$$y'(x) = \frac{1}{2}\cdot(C-x^2)^{-\frac{1}{2}}\cdot(-2x) = \frac{-x}{\sqrt{C-x^2}} = -\frac{x}{y}. \quad \square$$

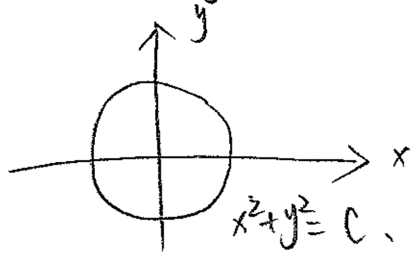
Defn. Suppose that solutions to the differential equation (△) or (✱) are given implicitly by the equation

$$F(x,y) = C.$$

Then the level sets defined by $F(x,y)=C$ are called **integral curves** of the differential equation.

Ex. Still same example as above.

$x^2+y^2 = C$ is the integral curve.



Ex. 1. $w = \sin(x+y)\,dx + (2y + \sin(x+y))\,dy$. exact.

$P = \sin(x+y)$, $Q = 2y + \sin(x+y)$.

$$\frac{\partial P}{\partial y} = \cos(x+y)$$

$$\frac{\partial Q}{\partial x} = \cos(x+y)$$ ✓.

2. $w = \frac{x}{\sqrt{x^2+y^2}}\,dx + \frac{y}{\sqrt{x^2+y^2}}\,dy$. exact.

$P = \frac{x}{\sqrt{x^2+y^2}}$, $Q = \frac{y}{\sqrt{x^2+y^2}}$.

$$\frac{\partial P}{\partial y} = -\frac{1}{2}x\,(x^2+y^2)^{-\frac{3}{2}} \cdot 2y = -xy\,(x^2+y^2)^{-\frac{3}{2}}$$

$$\frac{\partial Q}{\partial x} = -\frac{1}{2}y\,(x^2+y^2)^{-\frac{3}{2}} \cdot 2x = -xy\,(x^2+y^2)^{-\frac{3}{2}}.$$ ✓

3. $w = (2r + \ln y)\,dr + ry\,dy$. is not exact.

$P = 2r + \ln y$, $Q = ry$

$$\frac{\partial P}{\partial y} = \frac{1}{y}.$$

$$\frac{\partial Q}{\partial r} = y$$ ✓.

Solving exact differential equations.

If the equation $P(x,y)\,dx + Q(x,y)\,dy = 0$ is exact, the solution is given by $F(x,y) = C$ where F is found by solving $\frac{\partial F}{\partial x} = P$ and $\frac{\partial F}{\partial y} = Q$ using the steps:

1. Solve $\partial F/\partial x = P$ by integration:

$$F(x,y) = \int P(x,y)\,dx + \phi(y).$$

2. Solve $\partial F/\partial y = Q$ by choosing ϕ so that

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\int P(x,y)\,dx + \phi'(y) = Q(x,y).$$

Ex. $\sin(x+y)\,dx + (2y + \sin(x+y))\,dy = 0.$

Here $P(x,y) = \sin(x+y)$, $Q(x,y) = 2y + \sin(x+y)$

Step 1: $\partial F/\partial x = P(x,y) = \sin(x+y)$ by integration:

$$F(x,y) = \int \sin(x+y)\,dx + \phi(y)$$

$$= -\cos(x+y) + \phi(y)$$

Step 2: $\frac{\partial F}{\partial y} = \sin(x+y) + \phi'(y) = 2y + \sin(x+y) = Q(x,y)$

$\Rightarrow \phi'(y) = 2y \Rightarrow \phi(y) = y^2.$

So $F(x,y) = y^2 - \cos(x+y).$

Solutions.

Let's carefully examine what it means when we say the equation $F(x,y)=C$ gives the general solution to $\omega = P(x,y)\,dx + Q(x,y)\,dy = 0$.

Suppose $F(x,y) = C$. y is function of x.

do differential $\frac{d}{dx}$ on both sides.

$$\frac{d}{dx}F(x, y(x)) = 0$$

so $$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} = 0.$$

$$\Longrightarrow \quad \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$

Since $$P\,dx + Q\,dy = 0.$$

so $$\frac{\partial F}{\partial x}\Big/P = \frac{\partial F}{\partial y}\Big/Q = \mu = \mu(x,y).$$

Special Case: $\mu = 1$.

$$\omega = P(x,y)\,dx + Q(x,y)\,dy = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = dF.$$

called $\boxed{\text{differential of } F}$

Defn. A differential form $\omega = P(x,y)\,dx + Q(x,y)\,dy$ is $\boxed{\text{exact}}$ if it is the differential of a continuously differential function

i.e. $\exists\ F = F(x,y)$ s.t

$$P(x,y) = \frac{\partial F}{\partial x}, \quad Q(x,y) = \frac{\partial F}{\partial y}.$$

Q: 1. Given a differential form $\omega = P\,dx + Q\,dy$, how do we know if it is exact?

2. If ω is exact, is there a way to find F such that $dF = P\,dx + Q\,dy$?

Thm. For $\omega = P(x,y)\,dx + Q(x,y)\,dy$,

(a) If ω is exact, then $\partial P/\partial y = \partial Q/\partial x$.

(b) If $\partial P/\partial y = \partial Q/\partial x$ holds in a rectangle R, then ω is exact in R.

Pf: Only for (a). ω is exact $\Leftrightarrow$ $P = \partial F/\partial x$, $Q = \partial F/\partial y$

$$\frac{\partial P}{\partial y} = \partial(\partial F/\partial x)/\partial y = \frac{\partial^2 F}{\partial x \partial y}, \quad \frac{\partial Q}{\partial x} = \partial(\partial F/\partial y)/\partial x = \frac{\partial^2 F}{\partial x \partial y}.$$

so $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

for (b), see the book.

Here μ', μ are not depending on y,

we need $\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ also not depending on y.

so $\mu = e^{\int \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)dx}$ is one solution.

Let's summarize these results.

The form $Pdx + Qdy$ has an integrating factor depending on one of the variables under the following conditions:

- If $h = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ is a function of x only,

  then $\mu(x) = e^{\int h(x)dx}$ is an integrating factor.

(Another case)
- If $g = \frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ is a function of y only,

  then $\mu(y) = e^{-\int g(x)dy}$ is an integrating factor.

Ex. $\omega = xy\,dx + x^2 dy = 0$.

$P = xy \qquad Q = x^2$

$\frac{\partial P}{\partial y} = x.$

$\frac{\partial Q}{\partial x} = 2x.$

so ω is not exact.

Check: $h = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{x^2}(x - 2x) = -\frac{1}{x}$. only depends on x.

so $\mu(x) = e^{\int h(x)\,dx} = e^{-\ln|x|} = \frac{1}{|x|}$.

take $\mu(x) = \frac{1}{x}$.

so $\mu(x)(P\,dx + Q\,dy)$

$= \frac{1}{x}(xy\,dx + x^2\,dy)$

$= y\,dx + x\,dy$

$= d(xy)$.

so $f(x,y) = xy = C$.

so $y(x) = \frac{C}{x}$.

Homogeneous Equations

Defn. A function $G(x,y)$ is *homogeneous of degree* n if

$$G(tx, ty) = t^n G(x,y).$$

for all $t > 0$ and all $x \neq 0$, $y \neq 0$.

Ex. $\frac{1}{x^2+y^2}$ is homogeneous of degree -2.

$\ln(y/x)$ is homogeneous of degree 0.

$x - y - 2$ is not homogeneous.

Ex. $(x^2+y^2)\,dx - 2xy\,dy = 0$

Step 1: $P = x^2+y^2$, $Q = -2xy$

$$\frac{\partial P}{\partial y} = 2y, \quad \frac{\partial Q}{\partial x} = -2y.$$

how to find μ s.t. $\mu\left((x^2+y^2)dx - 2xy\,dy\right)$ is exact.

so we need to discuss more methods to find μ.

Integrating factors depending on only one variable:

In fact, we want to find μ such that

$\mu M = \mu P\,dx + \mu Q\,dy$ is exact.

i.e. $\frac{\partial}{\partial y}(\mu P) = \frac{\partial}{\partial x}(\mu Q)$. $(*)$

It is a partial differential equation for μ.

There is no procedure for solving this equation in general.

However, special case: μ only depend on one variable.

Like, μ does not depend on y, i.e. $\mu = \mu(x)$.

so $(*)$ becomes

$$\mu \frac{\partial P}{\partial y} = \mu \frac{\partial Q}{\partial x} + \mu' \cdot Q.$$

so $$\mu' = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\mu.$$

Solutions and integrating factors

Now μ is not necessarily 1.

$$\frac{\partial F}{\partial x} = \mu P, \quad \frac{\partial F}{\partial y} = \mu Q.$$

then $\mu[P\,dx + Q\,dy] = \mu P\,dx + \mu Q\,dy$

$$= \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = dF.$$

so $\mu[P\,dx + Q\,dy]$ is exact.

here μ is called $\boxed{\text{integrating factor}}$.

Now we have a strategy for solving $P\,dx + Q\,dy = 0$.

Step 1: Find an <u>integrating factor</u> μ,
so that $\mu P\,dx + \mu Q\,dy$ is exact.

Step 2: Find a function F such that
$$dF = \mu P\,dx + \mu Q\,dy.$$

then a general solution $y(x)$ to $P\,dx + Q\,dy = 0$
is given implicitly by $F(x, y) = C$.

But the 1st step is very difficult.

Repeat the steps for solving exact forms.

Step ①: $F(x,y) = \int \mu P(x,y)\,dx + \phi(y)$

$$= \int \left(1 + \frac{2x}{y}\right) dx + \phi(y)$$

$$= x + \frac{x^2}{y} + \phi(y).$$

Step ②: $\frac{\partial F}{\partial y} = -\frac{x^2}{y^2} + \phi'(y) = \mu Q(x,y) = -\frac{x^2}{y^2}.$

$$\Rightarrow \phi'(y) = 0$$

$$\Rightarrow \phi(y) \equiv C.$$

Step ③: $F(x,y) = x + \frac{x^2}{y} + C.$

so $x + \frac{x^2}{y} + C = C'$

$$\Rightarrow x + \frac{x^2}{y} = C$$

$$\Rightarrow \frac{x^2}{y} = C - x$$

$$\Rightarrow y = \frac{x^2}{C-x}. \quad \square.$$

Ex. $(y^2+2xy)\,dx - x^2\,dy = 0.$

Step 1: Check whether it is exact!

$P = y^2+2xy, \quad Q = -x^2$

$$\partial P/\partial y - \partial Q/\partial x = (2y+2x) - (-2x) = 2y+4x.$$

$$h = \frac{1}{Q}(\partial P/\partial y - \partial Q/\partial x) = -\frac{1}{x^2}(2y+4x) \quad \text{not good.}$$

$$g = \frac{1}{P}(\partial P/\partial y - \partial Q/\partial x) = \frac{1}{y^2+2xy}(2y+4x) = \frac{2}{y}$$

is a fcn only depending on y! $\boxed{\text{good!}}$

Step 2: Using the summary in the box,

use $\mu = e^{-\int g(y)\,dy} = e^{-\int \frac{2}{y}\,dy}$ integrating factor.

$$= e^{-2\ln|y|} = \frac{1}{y^2}.$$

Step 3: Multiply μ on both sides.

$$\Rightarrow \frac{1}{y^2}(y^2+2xy)\,dx - \frac{x^2}{y^2}\,dy = 0$$

$$\Rightarrow \left(1+\frac{2x}{y}\right)dx - \frac{x^2}{y^2}\,dy = 0.$$

exact.

Now repeat the steps for solving exact forms.

Step ①: $F(x,y) = \int \frac{1}{x}(xy-1)\,dx + \phi(y)$

$= \int (y - \frac{1}{x})\,dx + \phi(y)$

$= xy - \ln|x| + \phi(y)$.

Step ②: $\partial F/\partial y = x + \phi'(y) = \frac{1}{x}(x^2 - xy) = x - y$.

$\Rightarrow \phi'(y) = -y$

so $\phi(y) = -\frac{1}{2}y^2 + C$.

Step ③. so $F(x,y) = xy - \ln|x| + (-\frac{1}{2}y^2 + C)$.

so $xy - \ln|x| - \frac{1}{2}y^2 + C = C'$

so $xy - \ln|x| - \frac{1}{2}y^2 = C$.

$\Rightarrow y^2 - 2xy + 2\ln|x| + C = 0$

so $y = \dfrac{2x \pm \sqrt{4x^2 - 4(2\ln|x| + C)}}{2}$

$= x \pm \sqrt{x^2 - 2\ln|x| + C}$. □.

Ex. $(xy-1)\,dx + (x^2 - xy)\,dx = 0$

Step 1: Check whether it's exact.

$$P = xy - 1, \quad Q = x^2 - xy.$$

$$\partial P/\partial y = x, \quad \partial Q/\partial x = 2x - y.$$

It's not exact!

$$\partial P/\partial y - \partial Q/\partial x = -x + y.$$

$$h = \frac{1}{Q}(\partial P/\partial y - \partial Q/\partial x) = \frac{1}{x^2 - xy}(-x+y) = -\frac{1}{x}.$$

only depending on x. $\boxed{\text{Good!}}$

Step 2: Using the summary in the box:

use $\mu = e^{\int h(x)\,dx} = e^{\int -\frac{1}{x}\,dx}$

$$= e^{-\ln|x|} = \frac{1}{|x|}.$$

Integrating factor.

use $\mu = \frac{1}{x}$.

Step 3: Consider $\frac{1}{x}(xy-1)\,dx + \frac{1}{x}(x^2 - xy)\,dy$

which is exact.

HW 2:

§2.4. 5, 15, 19

§2.6. 16, 28

Extra: (1) Draw Direction Field of

$$x' = \frac{2tx}{1+x},$$

using Human Method.

(2) And sketch the integral curve of this direction field passing through $x(0) = 1$.

(3) Solve the initial value problem

$$x' = \frac{2tx}{1+x}, \quad x(0) = 1.$$

Draw the curve $x = x(t)$ on the tx-plane.

(Rmk: You can compare your answer in (3) with in (2).)