Lecture III.

Defn. A differential equation $Pdx + Qdy = 0$ is said to be homogeneous if P and Q are homogeneous of same degree n.

then we can introduce a new variable u with $y = xu$.

so $P(x, y) = P(x, xu) = x^n P(1, u)$

$Q(x, y) = Q(x, xu) = x^n Q(1, u)$.

$$\Rightarrow \cancel{x^n} P(1, u)\, dx + \cancel{x^n} Q(1, u)\, d(xu) = 0$$

$$\Rightarrow P(1, u)\, dx + Q(1, u)\, (u\,dx + x\,du) = 0$$

$$\Rightarrow \left(P(1,u) + Q(1,u)\, u\right) dx + Q(1, u)\, x\, du = 0.$$

$$\Rightarrow \frac{dx}{x} + \frac{Q(1,u)}{P(1,u) + u\,Q(1,u)}\, du = 0.$$

($f(x)\,dx + g(u)\,du = 0$ is called seperable equation.).

It is solvable, $\displaystyle \int \frac{dx}{x} = -\int \frac{Q(1,u)}{P(1,u) + u\,Q(1,u)}\, du$.

$$\Rightarrow \ln|x| = -\int \frac{Q(1,u)}{P(1,u) + u\,Q(1,u)}\, du$$

$$\Rightarrow |x| = e^{-\int \frac{Q(1,u)}{P(1,u)+u\,Q(1,u)}\, du} = k(u).$$

so $u = u(x)$ is implictly defined by an equation $k(u) = |x|$

so $y(x) = xu = x\,u(x)$.

Ex. $(x^2+y^2)\,dx - 2xy\,dy = 0.$

x^2+y^2, $-2xy$ are homogeneous of degree 2.

Step 1: use $y = xu$.

$\Rightarrow (x^2 + x^2u^2)\,dx - 2x\cdot xu\, d(xu) = 0$

$\Rightarrow \cancel{x^2}(1+u^2)\,dx - 2\cancel{x^2}u\,(u\,dx + x\,du) = 0$

$\Rightarrow (1+u^2-2u^2)\,dx - 2xu\,du = 0$

$\Rightarrow (1-u^2)\,dx - 2xu\,du = 0$

$\Rightarrow \frac{dx}{x} - \frac{2u}{1-u^2}\,du = 0$ ← seperable equation

$\Rightarrow \frac{dx}{x} = \frac{2u}{1-u^2}\,du$ ← seperate variables.

Step 2: Remember we want to solve for $y = y(x)$.

so first we solve for $u = u(x)$.

$\Rightarrow \frac{2v}{1-v^2}\,dv = \frac{dx}{x}$ Integrate.

$\Rightarrow \int \frac{2v}{1-v^2}\,dv = \int \frac{dx}{x}$

$\Rightarrow |1-u^2| = \frac{e^c}{|x|}$

Step 3: Solve it $\Rightarrow (1-u^2) = \frac{c}{x} \Rightarrow u = u(x) = \pm\sqrt{1-\frac{c}{x}}$

so $y(x) = xu = \pm x\sqrt{1-\frac{C}{x}}$ is solution.

Ex. $(y + 2x e^{-y/x}) dx - x dy = 0.$

First check $y + 2xe^{-y/x}$, $-x$ are homogeneous of degree 1.

Step 1: use $y = xv$.

$\Rightarrow (xv + 2xe^{-v})\, dx - x\, d(xv) = 0$

$\Rightarrow (xv + 2xe^{-v})\, dx - x\, dx \cdot v - x \cdot x\, dv = 0$

$\Rightarrow 2xe^{-v} dx - x^2 dv = 0$

$\Rightarrow 2e^{-v} dx - x\, dv = 0 \leftarrow$ seperable equation

$\Rightarrow \frac{1}{2} e^{v} dv = \frac{dx}{x}$ $\leftarrow$ seperate variable.

Step 2: Integrate it:

Since we want to solve for $y = y(x)$, here we plane first solve for $v = v(x)$.

$$\int \frac{1}{2} e^{v} dv = \int \frac{dx}{x}$$

$\Rightarrow \frac{1}{2} e^{v} = \ln|x| + C$

$\Rightarrow v = \ln(2\ln|x| + C)$

Step 3: $y = xv = x \ln(2\ln|x| + C).$

□.

Existence of solutions.

In most cases, we cannot solve differential equations explicitly. But we still have some methods for studying properties of solutions even when we dont know the solutions explicitly. like existence and uniqueness.

Ex. Consider the initial value problem

$$tx' = x + 3t^2, \text{ with } x(0) = 1.$$

If we plug in $x(0)=1$, i.e $t=0$, $x=1$.

so $0 \cdot x' = 1 + 3 \cdot 0^2 \Rightarrow 0 \cdot x' = 1$. $\not\exists$.

Now we give conditions of "existence of solutions".

Thm. (Existence of Solutions)

Suppose the function $f(t,x)$ is defined and continuous on the rectangle R in the tx-plane. Then given any point $(t_0, x_0) \in R$, the initial value problem $x' = f(t,x)$ and $x(t_0) = x_0$ has a solution $x(t)$ defined in an interval containing t_0. Furthermore, the solution will be defined at least until the solution curve $t \longrightarrow (t, x(t))$ leaves the rectangle R.

Ex. $tx' = x + 3t^2$ w/ $x(0) = 1$.

$\Rightarrow x' = \frac{x}{t} + 3t = f(t,x)$.

then $f(t,x)$ is not continuous at $t=0$. not satisfy condition.

so we could not apply "existence theorem".

hence the result of example does not contradict with theorem.

The Interval of Existence of a Solution.

Defn. The interval of existence of a solution is the largest interval in which the solution can be defined.

Ex. $\frac{dy}{dt} = \frac{t}{y+1}$, $y(2) = 0$.

Find the interval of existence.

Firstly, Find an explicit solution.

Step 1: Seperate Variables.

$(y+1)\,dy = t\,dt$.

Step 2: Integrate it:

$$\int (y+1)\,dy = \int t\,dt$$

$$\Rightarrow \frac{1}{2}y^2 + y = \frac{1}{2}t^2 + C.$$

$$\Rightarrow \quad y^2+2y-(t^2+C)=0$$

$$\Rightarrow y = \frac{-2 \pm \sqrt{4+4(t^2+C)}}{2}$$

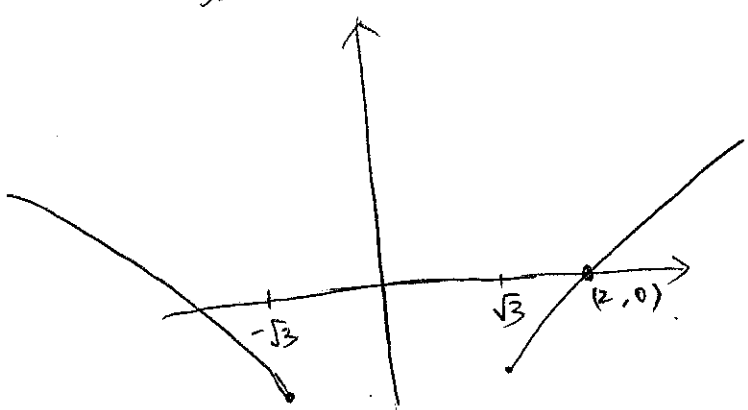
$$= -1 \pm \sqrt{t^2+C} \, .$$

Since $y(2)=0$. $\Rightarrow \begin{cases} \text{only } + \text{ sign is solution.} \\ -1+\sqrt{2^2+C}=0 \Rightarrow C=-3. \end{cases}$

So $y=-1+\sqrt{t^2-3}$ is our solution.

Secondly. Now need $t^2-3 \geq 0 \Rightarrow t \geq \sqrt{3}$ or $t \leq -\sqrt{3}$.

So



since our solution satisfying $y(2)=0$.

the right part is our solution curve.

so the interval of existence is $[\sqrt{3}, \infty)$.

Ex. Existence for Linear Equations.

$x' = a(t)x + f(t)$.

If $a(t)$ and $f(t)$ are continuous on the interval $b<t<C$.

Consider $R = \{ b<t<C, -\infty < x < \infty \}$.

In this case, a $\boxed{\text{stronger existence}}$ thm can be proved, which guarantees that solutions exist over the entire interval $b<t<C$.

so if $a(t)$, $f(t)$ are continuous for all t,

then the interval of existence of a solution

to $x' = a(t)x + f(t)$

is $(-\infty, \infty)$.

Ex. $\frac{dy}{dt} = \frac{1}{(t+2)(y-3)}$, $y(0) = 1$.

Firstly, solve it.

Step 1: seperate variables.

$$(y-3)\,dy = \frac{dt}{t+2}$$

Step 2: Integrate it:

$$\int (y-3)\,dy = \int \frac{dt}{t+2}.$$

$$\Rightarrow \frac{1}{2}y^2 - 3y = \ln|t+2| + C$$

$$\Rightarrow y^2 - 6y = 2\ln|t+2| + C.$$

Step 3: Solve $y = y(t)$.

$$y = \frac{6 \pm \sqrt{36 - 4(2\ln|t+2| + C)}}{2}$$

$$= 3 \pm \sqrt{C - 2\ln|t+2|}.$$

Step 4: Solve for C.

since $y(0) = 1 < 3$.

$$\Rightarrow \begin{cases} ① & - \text{ sign } \checkmark \\ ② & 3 - \sqrt{C - 2\ln 2} = 1 \Rightarrow C = 2\ln 2 + 4. \end{cases}$$

so $y = 3 - \sqrt{2\ln 2 + 4 - 2\ln|t+2|}$.

Secondly. Analyze the solution.

we need $2\ln 2 + 4 - 2\ln|t+2| \geq 0$.

$\Rightarrow \quad \ln|t+2| \leq \ln 2 + 2$

$\Rightarrow \quad |t+2| \leq e^{\ln 2+2} = 2e^2$

$\Rightarrow \quad -2e^2 - 2 \leq t \leq 2e^2 - 2$

since $0 \in (-2e^2-2, 2e^2-2)$.

so the interval of existence is

$(-2e^2-2, 2e^2-2)$.

For uniqueness of solutions, we need more conditions.

We can see the example here.

Consider the initial value problem

$$x' = x^{\frac{1}{3}}, \quad \text{with } x(0)=0.$$

First we can guess one solution is $x(t) = 0$.

And in fact we have another solution:

$$\frac{dx}{dt} = x^{\frac{1}{3}}$$

$$\Rightarrow \frac{dx}{x^{\frac{1}{3}}} = dt$$

$$\Rightarrow \int \frac{dx}{x^{\frac{1}{3}}} = \int dt$$

$$\Rightarrow \frac{3}{2} x^{\frac{2}{3}} = t + C.$$

$$\Rightarrow x = \left(\frac{2}{3}(t+C)\right)^{\frac{3}{2}}.$$

since $x(0)=0 \Rightarrow C=0$.

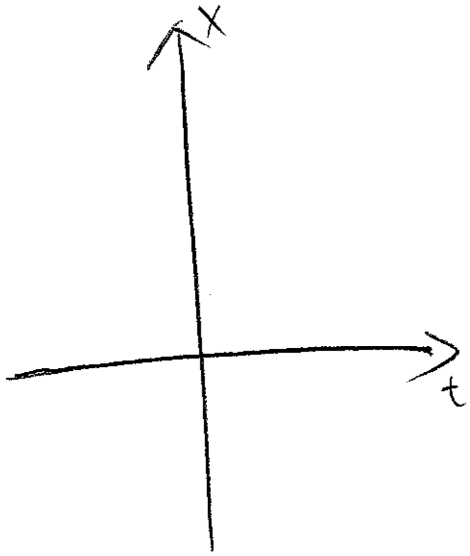
so $x(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}}, \ t>0.$

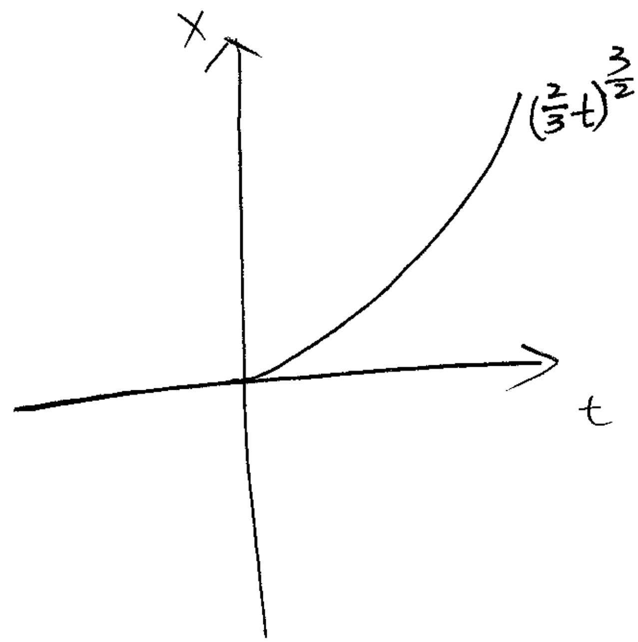
so

$$x(t) = \begin{cases} \left(\frac{2}{3}t\right)^{\frac{3}{2}}, & t>0 \\ 0, & t \le 0 \end{cases}$$

is another solution.

so we see





are both solution curves of our initial value problem.

But you see, our problem satisfy conditions in "Existence of solution" - thm.

so we need more conditions to get "uniqueness of solution".

Uniqueness of Solutions.

Thm. (Uniqueness of Solutions)

Suppose that the fcn $f(t,x)$ and its partial derivative $\partial f/\partial x$ are both continuous on the rectangle R in the tx-plane.

Suppose $(t_0, x_0) \in R$ and that the solutions

$$x' = f(t,x) \quad \text{and} \quad y' = f(t,y)$$

satisfying $x(t_0) = y(t_0) = x_0$.

Then as long as $(t, x(t))$ and $(t, y(t))$ stay in R,

we have $x(t) = y(t)$.

(i.e. the solution in R is unique.)

So our strategy for proving solution's uniqueness is as followings:

Step 1: Check ① $f(t, y)$ is continuous in R.

② $\partial f / \partial y$ is continuous in R.

③ (x_0, y_0) is in the rectangle R.

Step 2: Get conclusion.

If ①, ②, ③ hold,

applying "uniqueness of solution" - thm,

we have the solution is unique.

Geometric Interpretation of Uniqueness.

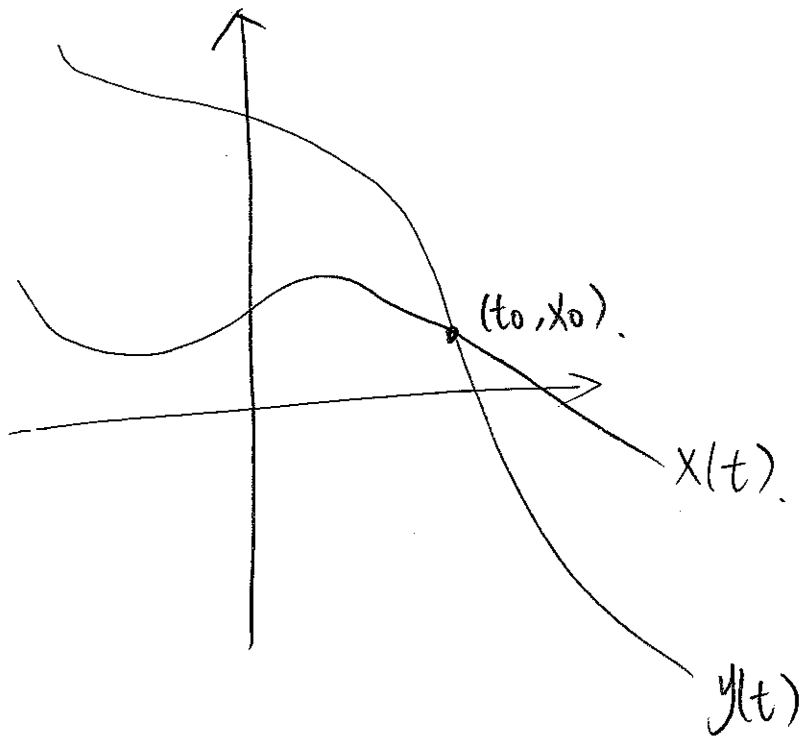
Let's look at the graph of the solutions — the solution curves.

If we have two fcns $x(t)$ and $y(t)$ that satisfy $x(t_0) = y(t_0) = x_0$ for some t_0, then the graphs of $x(t)$ and $y(t)$ meet at the pt (x_0, y_0).

If in addition we know $x(t)$ and $y(t)$ are solutions to the same ODE and satisfying conditions of "uniqueness of solutions"-thm then the thm implies that $x(t) = y(t)$ for all t.

In other words, the graphs of $x(t)$ and $y(t)$ coincide.

i.e. The two distinct solution curves never meet.



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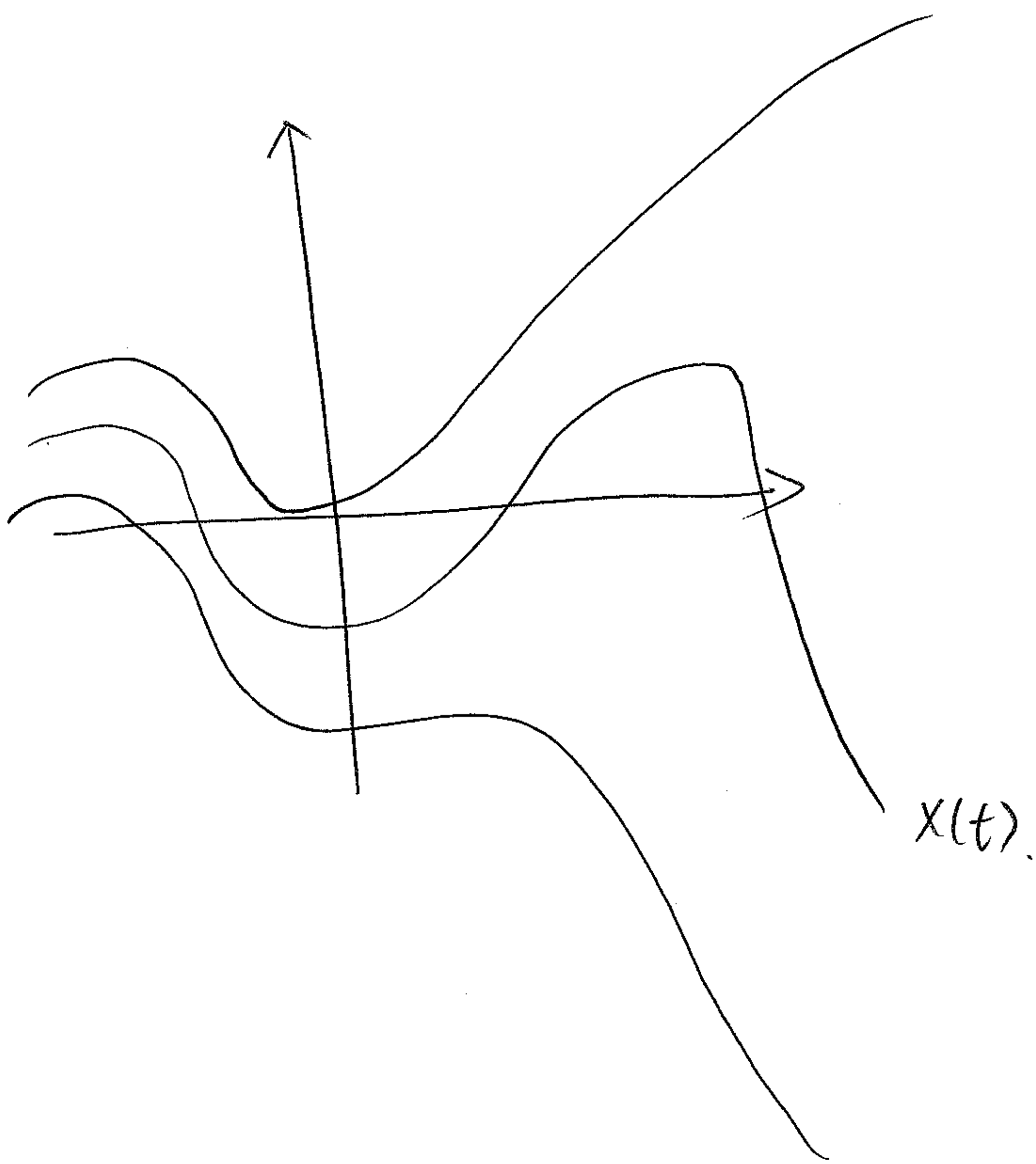


Applying the existence and uniqueness thm. (I)

We usually want to apply the thms to a special initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0.$$

To be able to apply the thm, it is necessary to find a rectangle R in which the equation satisfies the hypotheses.

Ex. Consider $tx' = x + 3t^2$. Is there a solution to this equation with the initial condition $x(1) = 2$? If so, is the solution unique?

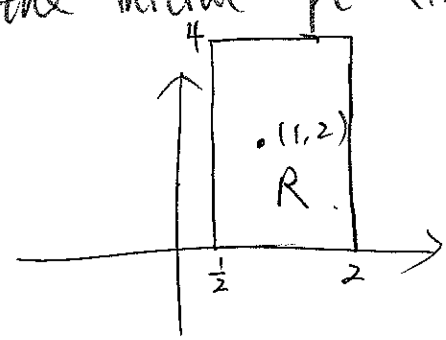
Step 0. To begin with, we write the equation in normal form.

$$x' = \frac{x}{t} + 3t.$$

so the right side, $f(x,t) = (x/t) + 3t$, is continuous except where $t = 0$.

We can take R to be any rectangle which contains the initial pt $(1, 2)$ and avoids $t = 0$.

like:



Then f is continuous everywhere in R.

so the conditions of "existence of solution" - thm are satisfied.

We can conclude that there is a solution to the initial value problem.

Furthermore, to apply "uniqueness of solution" - thm,

· Check ①: $f(t,x)$ is continuous. in R.

②: $\partial f/\partial x = \frac{1}{t}$ is continuous in R.

③: $(1,2) \in R$.

All conditions in thm are satisfied.

$\Rightarrow$ there is only one solution to IVP.

Apply the existence and Uniqueness theorems (II)

Ex. Consider $x' = (x-1)\cos(xt)$.

and suppose we have a solution $x(t)$ that satisfies $x(0) = 1$.

We claim that $x(t) = 1$ for all t. How do we prove it?

Pf: First, $x(t) = 1$ for all t is a solution of the ODE passing through $x(0)=1$.

Left side $= 0$, Right side $= (1-1)\cos t = 0$.

Second: The solution of $x' = (x-1)\cos(xt)$ is unique.

That's because:

Check ①: $f(t,x) = (x-1)\cos(xt)$ is continuous on xy-plane

②: $\partial f/\partial x = \cos(xt) + (x-1)(-\sin(xt))\cdot t$

$= \cos(xt) - (x-1)t\sin(xt)$ is continuous on xy-plane.

by applying "Uniqueness of solutions" – thm,

The solution is unique.

Conclusion: $x(t) = 1$ for all t. $\square$.

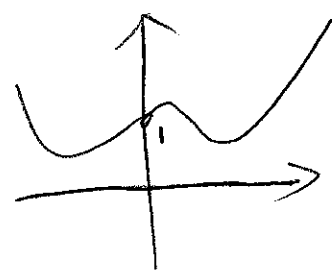
Ex. Suppose that x is a solution to the initial value problem

$$x' = x\cos^2 t \quad \text{and} \quad x(0) = 1.$$

Show that $x(t) > 0$ for all t for which x is defined.

Pf: Firstly, $x_0 = x_0(t) \equiv 0$ is a solution to $x' = x\cos^2 t$.
(Step 1) because left side $= 0$
right side $= 0 \cdot \cos^2 t = 0$.

Secondly, if the solution is unique.
(Step 2) and $x(0) = 1$ is above $x_0 = 0$.



so $x(t)$ passing through $x(0) = 1$
is also above $x_0 \equiv 0$ for all t.
$\Rightarrow$ $x(t) > 0$ for all t for which x is defined.

Thirdly, To prove the solution is unique
(Step 3). by applying "uniqueness of solution" - thm.

Check ①: $f(t, x) = x\cos^2 t$ is continuous.

②: $\partial f / \partial x = \cos^2 t$ is continuous.

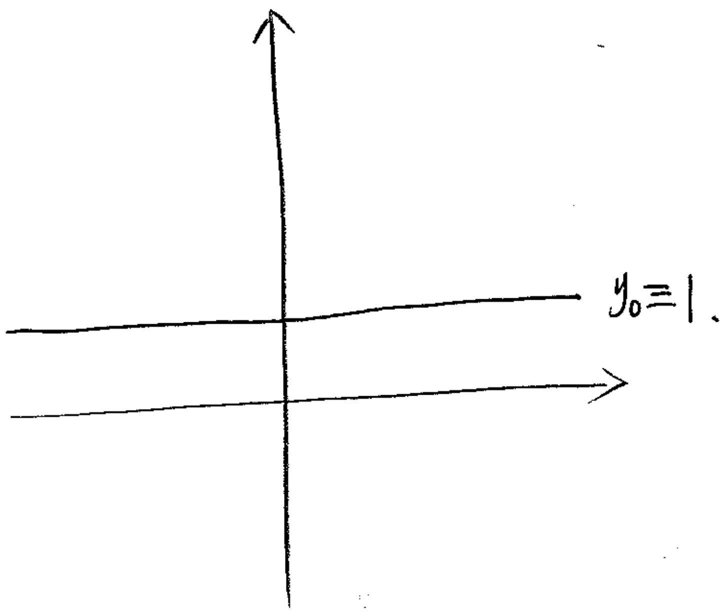
$\Rightarrow$ the solution is unique.

$\square$.

Ex. $y' = (y-1)\sin t$. . .

We know $y_0 \equiv 1$ is a solution to this ODE.

left side $= 0$, right side $= (1-1)\cos t = 0$.



And check ①: $f(t,y)$ is continuous.

② : $\partial f/\partial y = \sin t$ is continuous.

$\Rightarrow$ can apply "uniqueness of solutions" – thm.

so solution is unique.

$\xrightarrow[\text{View}]{\text{Geometric}}$ Every 2 solution curves $y_1(t)$ and $y_2(t)$ never intersect.

Since we've already known $y_0 \equiv 1$ is a solution curve.

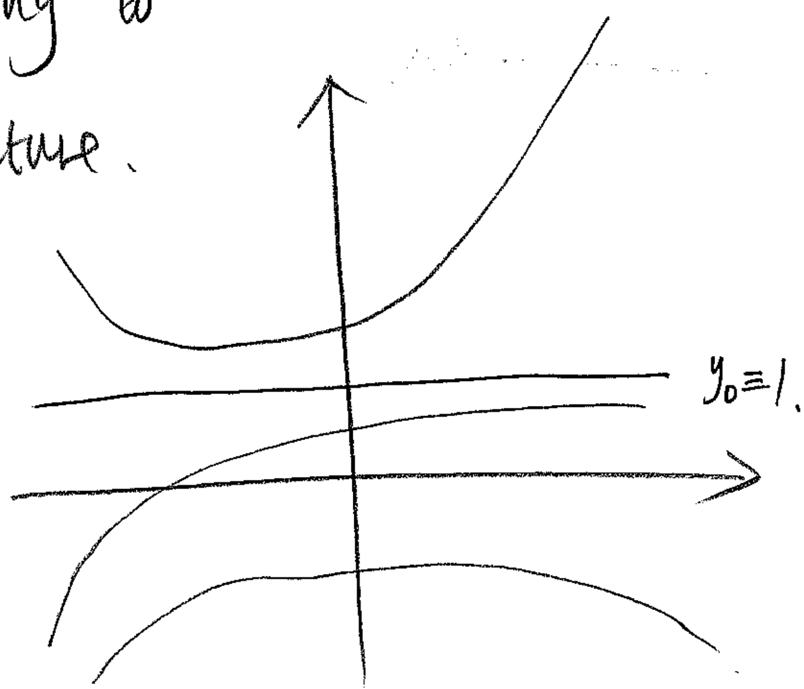
Every other solution curve will never cross $y_0 \equiv 1$.

So

Either $y(t) > 1$ for all t where y is defined

Or $y(t) < 1$ for all t where y is defined.

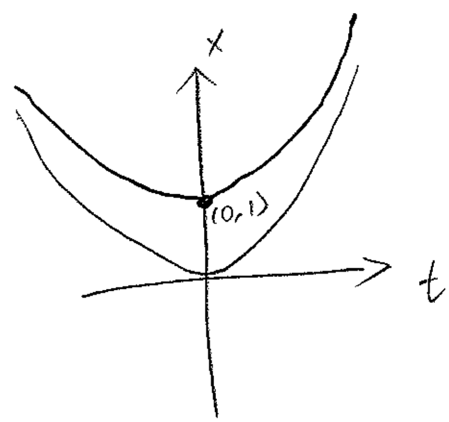
Corresponding to the picture.



Ex. Suppose that x is a solution to the initial value problem $x' = x - t^2 + 2t$ and $x(0) = 1$. Show that $x(t) > t^2$ for all t for which x is defined.

Pf: Firstly, $x_0(t) = t^2$ is a solution to

$$x' = x - t^2 + 2t.$$



Left side $= 2t$

Right side $= t^2 - t^2 + 2t = 2t$.

Secondly, our goal is to prove uniqueness of solution.

If so, since our solution passing through $x(0) = 1$, the graph of it should be above $x_0(t) = t^2$.

So our solution $x(t) > x_0(t) = t^2$.

Thirdly. To prove uniqueness of solution is to apply "uniqueness of solution" - thm.

Check ①: $f(t, x) = x - t^2 + 2t$ is continuous.

②: $\partial f / \partial x = 1$ is continuous.

$\Rightarrow$ the solution is unique.

$\square$.

Ex. Suppose that y is a solution to the initial value problem

$$y' = (y^2-1)e^{ty} \quad \text{and} \quad y(1)=0.$$

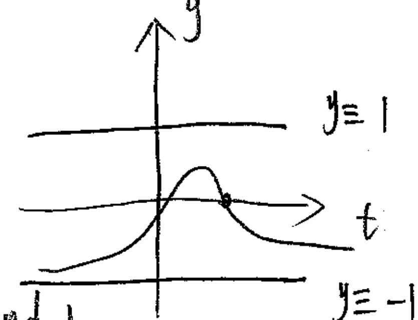
Show that $-1 < y(t) < 1$ for all t for which y is defined.

Pf: Step 1: $y_1 \equiv 1$, $y_2 \equiv -1$ are both solutions to $y' = (y^2-1)e^{ty}$.

Step 2: If the solution is unique.

Since $-1 \equiv y_2 < 0 < y_1 \equiv 1$.

so the solution passing through $y(1)=0$ is between -1 and 1.



$\Rightarrow -1 < y \doteq y(x) < 1$.

Step 3: To prove the solution's uniqueness, by applying "uniqueness of solution" - thm.

Check: ① $f(t,y) = (y^2-1)e^{ty}$ is continuous.

② $\partial f / \partial y = 2ye^{ty} + (y^2-1)te^{ty}$ is continuous.

so the solution is unique. □