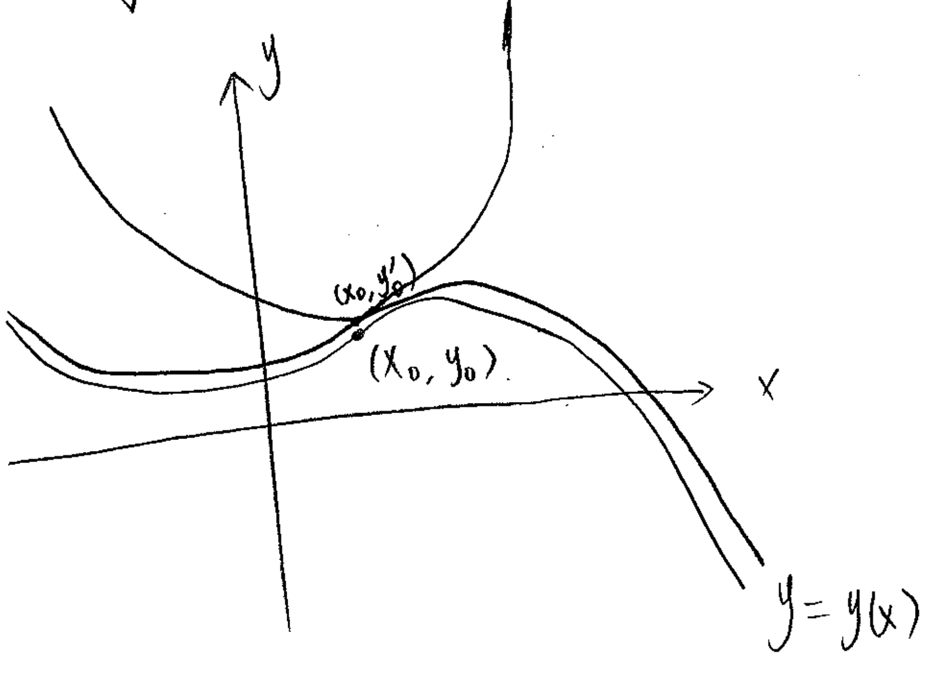
Lecture IV

Dependence of Solutions on Initial Conditions

Motivation.

Given that we have an error in the initial conditions, is the solution still close enough to the real solution?

$\Downarrow$ Geometric View



So our question is just for a given ODE, Which one of the red and the blue happens?

But how to measure "close" between 2 curves?

there is one measurement.

For every t, consider $|x(t) - y(t)|$.

Thm. Suppose the fcn $f(t,x)$ and $\partial f/\partial x$ are both continuous on the rectangle R in tx-plane and let $M = \max_{(t,x)\in R} \left|\frac{\partial f}{\partial x}(t,x)\right|$.

Suppose (t_0, x_0) and (t_0, y_0) are in R and that

$$x'(t) = f(t, x(t)) \text{ and } x(t_0) = x_0$$

$$y'(t) = f(t, y(t)) \text{ and } y(t_0) = y_0.$$

Then as long as $(x, x(t))$ and $(t, y(t))$ belong to R, we have

$$|x(t) - y(t)| \le |x_0 - y_0| \, e^{M|t - t_0|}.$$

Ex. $x' = x\cos t$.

Suppose $x_1 = x_1(t)$ is solution to ODE w/ initial condition $x_1(0) = 2$

Suppose $x_2 = x_2(t)$ is solution of ODE w/ initial condition $x_2(0) = 3$.

Q: Can we get a bound for $|x_1(10) - x_2(10)|$?

A: The only way to get the bound is trying to apply the thm.

Step 1: Check for the conditions.

① $f(t, x) = x\cos t$ is continuous.

② $\partial f/\partial x = \cos t$ is continuous.

so can apply the thm.

Step 2: Give a reasonable $R = \{0 \le t \le 10, -10^6 \le x \le 10^6\}$.

the interval for t is reasonable.

What does Thm tell us?

① If $|x_0 - y_0|$ is smaller, i.e. (t_0, x_0) and (t_0, y_0) are closer,

then $|x(t) - y(t)|$ is smaller.

② If $|t - t_0|$ is smaller,

then $|x(t) - y(t)|$ is smaller.

③ If R is a bounded rectangle $\{a<t<b,\ c<y<d\}$.

then $|t - t_0|$ is bounded by $b-a$.

And M is bounded, because $\partial f/\partial x$ is continuous

and R is bounded.

hence $|x(t) - y(t)| \leq$ bounded value.

so we get a control of the difference.

If we translation in our motivation question,

that means if ODE satisfies conditions in thm,

then we can get a control of the solution and the real one.

difference between

For x. we need $(t, x(t))$ for $0 \le t \le 10$ could not go out R, just choose enough interval for x.

Since $|\partial f/\partial x| = |\cos t| \le 1$.

so $M = \max_{R} |\partial f/\partial x| = 1$.

hence
$$|x_1(10) - x_2(10)| \le |x_1(0) - x_2(0)| e^{M|10-0|}$$
$$= |2-3| e^{1\cdot 10}$$
$$= e^{10}.$$

Mixing Problems. (Applications)

Ex. The 600-gal tank is filled with 300 gal of pure water. A spigot is opened above the tank and a salt solution containing 1.5 lb of salt per gallon of solution begins flowing into the tank at a rate of 3 gal/min. Simultaneously, a drain is opened at the bottom of the tank allowing the solution to leave the tank at a rate of 1 gal/min. What will be the salt content in the tank at the precise moment that the volume of solution in the tank is equal to the tank's capacity (600 gal)?

A: Since the volume of solution inceases at a net rate of 2 gal/min
$= 3$ gal/min $- 1$ gal/min.

so the volume of solution at time t,
is given by $V(t) = 300 + 2t$.

The rate at which the salt enters the tank given by

rate in $= 3$ gal/min $\times$ 1.5 lb/gal $= 4.5$ lb/min.

Assume perfect mixing, the concentration of the solution in the tank is

$$C(t) = \frac{x(t)}{V(t)} = \frac{x(t)}{300 + 2t} \text{ lb/gal}.$$

Therefore, the rate at which salt leaves through the drain at the bottom of the tank is

$$\text{rate out} = 1 \text{ gal/min} \times \frac{x(t)}{300+2t} \text{ lb/gal} = \frac{x(t)}{300+2t} \text{ lb/min}.$$

The balance law now yields

$$\frac{dx}{dt} = \text{rate in} - \text{rate out},$$

$$\Rightarrow \quad x' = 4.5 - \frac{x(t)}{300+2t}.$$

Can we solve the equation?

Yes, linear differential equation.

$$\Rightarrow x(t) = 450 + 3t + \frac{C}{\sqrt{300+2t}}.$$

Solve for C. we need initial condition.

Since at the beginning, the tank is filled with pure water initially, so the initial salt content is zero.

Thus $x(0) = 0$ and $0 = x(0) = 450 + \frac{C}{\sqrt{300}}$

$$\Rightarrow C = -4500\sqrt{3}.$$

$$\Rightarrow x(t) = 450 + 3t - \frac{4500\sqrt{3}}{\sqrt{300+2t}}.$$

At time $t = 2.5$ hour $= 150$ min, $x(t) = 450 + 3 \times 150 - \frac{4500\sqrt{3}}{\sqrt{300+2 \cdot 150}} \approx 582$ lb.

The logistic Model of growth.

Whenever we have a mathematical model, we should examine its implications carefully to discover to what extent the model correctly predicts what happens in the real world.

In the case of the Malthusian model the prediction of unlimited exponential growth is clearly impossible.

If a colony of protozoa grew exponentially, it would cover the earth in times that are observable, and this just does not happen.

Check our assumptions, can see the problem.

We have assumed the colony has no lack of nutrients and no lack of space in which to grow.

But this is not true in real life, we need to take this into account.

so the death rate should be $d + aP$.

the birth rate should be $b - cP$.

hence $$\frac{dP}{dt} = (d+aP-(b-cP))P$$
$$= ((d-b)+(a+c)P)P$$

Set $r_0 = b-d$, $r_0/K = a+c$.

called natural reproductive rate.

so
$$\boxed{\begin{aligned} P' &= (r_0 - (r_0/K)P)\,P \\ &= r_0(1-P/K)P. \end{aligned}}$$

called logistic equation.

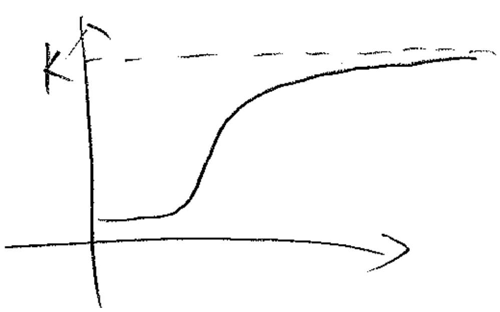
And the model of population growth embodied in the logistic equation is called the logistic model.

We can solve for it.

$$P(t) = \frac{KP_0}{P_0 + (K-P)\,e^{-r(t-t_0)}}.$$

Analyse it, as $t \to \infty$, $P(t) \longrightarrow \frac{KP_0}{P_0} = K$.

so K is stable part



called the carrying capacity.

$10 = P(0) = C$

$25 = P(1) = 10e^{r}$

$\Rightarrow \quad r = \ln(2.5) \approx 0.9163.$

so $\quad P(10) = 10e^{r \times 10} \approx 95.367. \qquad \square.$

Ex. Consider a model of growth of populations of ... (like, birds).

Assume b is birth rate, d is death rate,

and $P(t)$ is the population at time t.

so $$P'(t) = b\,P(t) - d\,P(t)$$

$$= (b-d)\,P(t).$$

$$\Rightarrow P(t) = C\cdot e^{(b-d)t}$$

$r = b - d$ is called reproductive rate.

This model is the Malthusian model.

Ex. A biologist start with 10 cells in a culture. Exactly 24 hours later he counts 25. Assuming a Malthusian model. What is the reproductive rate? What will be the number of cells at the end of ten days?

Matrix Algebra.

Vector and Matrices.

Defn. A matrix is of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m\times n}.$$

And we say A is a matrix with m rows and n columns.

a_{ij} is called entry.

Ex. $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}_{2\times 2}$ $\begin{pmatrix} 3 \\ 1 \end{pmatrix}_{2\times 1}$, $(1, 2)_{1\times 2}$, $(10)_{1\times 1}$.

$$\begin{pmatrix} 2 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}_{2\times 3}.$$

Defn. In the definition above,

let $a_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}$, $a_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}$, $\ldots$, $a_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$

denote the $\boxed{\text{column vectors}}$ in the matrix A.

Similarly, let

$$b_1 = (a_{11}\ a_{12}\ a_{13}\ \cdots\ a_{1n})$$

$$b_2 = (a_{21}\ a_{22}\ a_{23}\ \cdots\ a_{2n})$$

$$\vdots$$

$$b_m = (a_{m1}\ a_{m2}\ a_{m3}\ \cdots\ a_{mn})$$

are called row vectors of A.

Ex. for $A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$.

the row vectors are $b_1 = (1\ 2)$

$b_2 = (2\ 2)$

the column vectors are $a_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $a_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

Matrix Products.

Ex. $\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1\times 2 + 2\times 1 = 4 & 1\times 1 + 2\times 1 = 3 \\ 2\times 2 + 2\times 1 = 6 & 2\times 1 + 2\times 1 = 4 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 6 & 4 \end{pmatrix}$$

Ex. $\begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 1\times1+2\times1+3\times1 = 6 \\ 5\times1+6\times1+7\times1 = 18 \end{pmatrix} = \begin{pmatrix} 6 \\ 18 \end{pmatrix}.$$

Ex. $\begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$

cannot multiply.

so conclusion:

If matrices A and B can do product,
we need # of olumn of A = # of rows of B.

Ex. $\begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$? ×

$\begin{pmatrix} 2 & 3 & 2 & 1 \\ 1 & 2 & 5 & 6 \\ 7 & 8 & 9 & 10 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$? ×

$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \end{pmatrix}$? ✓

How about
For a number k, $k \cdot A = k \cdot \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$

$$\triangleq \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix}.$$

Ex. $2 \cdot \begin{pmatrix} 0.1 & 2 \\ 3 & 0.5 \end{pmatrix} =$

Ex. $5 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} =$

How about $A + B$?

Ex. $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 5 & 4 \\ 2 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 2+5=7 & 1+4=5 \\ 1+2=3 & 2+3=5 \end{pmatrix} = \begin{pmatrix} 7 & 5 \\ 3 & 5 \end{pmatrix}.$$

Ex. $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+3=4 \\ 2+1=3 \\ 3+2=5 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}$.

Ex. $\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} =$ X.

cannot plus.

Ex. $\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix} =$ X.

cannot plus.

so only matrice of $\underbrace{\text{same type}}_{m \times n}$ can do plus operation.

Now we do more example.

Ex. $3\left(\begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 5 & 1 \\ 4 & 2 & 4 \end{pmatrix}\right)$

$= 3\begin{pmatrix} 5 & 8 & 2 \\ 5 & 3 & 6 \end{pmatrix} = \begin{pmatrix} 15 & 24 & 6 \\ 15 & 9 & 18 \end{pmatrix}$.

$$3\begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix} + 3\begin{pmatrix} 3 & 5 & 1 \\ 4 & 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 9 & 3 \\ 3 & 3 & 6 \end{pmatrix} + \begin{pmatrix} 9 & 15 & 3 \\ 12 & 6 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} 15 & 24 & 6 \\ 15 & 9 & 18 \end{pmatrix}.$$

so we get

$$3\left(\begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 5 & 1 \\ 4 & 2 & 4 \end{pmatrix}\right) = 3\begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix} + 3\begin{pmatrix} 3 & 5 & 1 \\ 4 & 2 & 4 \end{pmatrix}.$$

Write this as a fact.

Fact. $k(A+B) = kA + kB$.

Ex. $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix}$

how do we do calculation?

① $$\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix}\right)\begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & 3 \\ 9 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 68 \\ 63 \end{pmatrix}.$$

② $$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\left(\begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 21 \\ 26 \end{pmatrix} = \begin{pmatrix} 68 \\ 63 \end{pmatrix}.$$

so we know both ways end same result.

Write this in a fact.

Fact. $A(BC) = (AB)C.$

Thm. Suppose A, B and C are matrices.

Then, assuming that the sizes of A, B and C allow the products to be defined, we have the following:

1. Multiplicative is associative: $A(BC) = (AB)C$.
2. Multiplication is distributive:

$$A(B+C) = AB + AC.$$

$$(B+C)A = BA + CA.$$

Now we check another important thing.

Consider $A = \begin{pmatrix} 0 & 1 \\ 9 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & -1 \\ 1 & 2 \end{pmatrix}$.

$$AB = \begin{pmatrix} 1 & 2 \\ -25 & -5 \end{pmatrix} \quad \text{and} \quad BA = \begin{pmatrix} -9 & -5 \\ 18 & 5 \end{pmatrix}.$$

so $AB \neq BA$.

so we see that the matrices A and B do not commute.

So the noncommutativity is the essential difference between matrices and numbers.

Rmk: You may think this is not so nice thing.

But not!

Because in real life a lot of things are not commutative. like first do transformation A and then B, the result is different from first do transformation B and then A.

So that makes matrices can represent our real world more precisely.

That is why physical use a lot of matrices.

Identity matrix

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{n\times n}.$$

The special thing of Identity matrix.

$$I \cdot A = A = A \cdot I. \quad \forall A \text{ allowed to product.}$$

Check: $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 2\times 1 + 1\times 0 & 2\times 0 + 1\times 1 \\ 1\times 1 + 1\times 0 & 1\times 0 + 1\times 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The transpose of a matrix

Defn. Suppose $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$,

then the transpose of A is $A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$

Ex.

$$A = \begin{pmatrix} 0 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix}.$$

$$\Rightarrow A^T = \begin{pmatrix} 0 & -2 \\ 1 & 2 \\ 2 & 3 \end{pmatrix}.$$

Ex. $A = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ \vdots \\ n \end{pmatrix}$

$$\Rightarrow A^T = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \end{pmatrix}.$$

Application.

$\begin{cases} 3x + 4y = 9 \\ 5x + y = 10 \end{cases}$ can be written in the form

$$\begin{pmatrix} 3 & 4 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 10 \end{pmatrix}.$$

so $\boxed{\text{solving linear equation systems}} \Longleftrightarrow \boxed{\text{vector solution } v \text{ s.t } Av = B.}$

Ex. $x_1 + 3x_2 - x_3 = 2$

$-2x_1 + 3x_2 - 2x_3 = -3$

$2x_1 - x_3 = 0.$

$$\Longleftrightarrow \begin{pmatrix} 1 & 3 & -1 \\ -2 & 3 & -2 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}.$$