Lecture V

One linear equation in two variable.

Consider the equation $3x+4y=9$. (1)

The set of all vectors $(x,y)^T$ that solve this equation will be referred to as the solution set of (1)

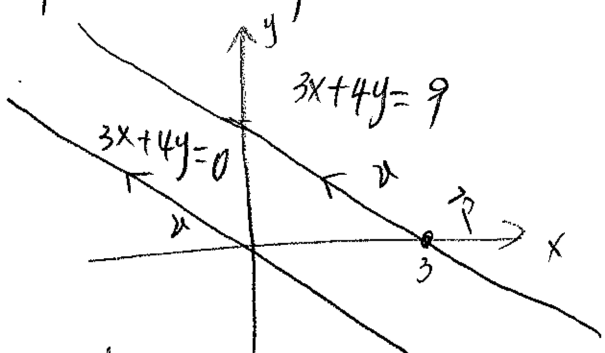
In this case, the solution set is a line in the plane.

If we solve (1) for x, getting $x = (9-4y)/3$,

then the vectors in the solution set have the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (9-4y)/3 \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + y\begin{pmatrix} -\frac{4}{3} \\ 1 \end{pmatrix}. \qquad (2)$$

Equation (2) is a parametric representation for the line defined by (1).



When $y=0$, we see that

$\vec{p} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ is a pt on the line.

The other pts on the line are obtained by starting at $\vec{p}$ and adding arbitrary multiples of the vector $v = \begin{pmatrix} -\frac{4}{3} \\ 1 \end{pmatrix}$.

Defn. A system of equations $Ax=b$ is said to be homogeneous if $b=0$. If $b \neq 0$, the system is inhomogeneous.

Rmk: homogeneous solutions always passing through origin.

Defn. A line in $\mathbb{R}^n$ is a set of the form

$$\{x = \vec{p} + t\vec{v} \mid t \in \mathbb{R}\}.$$

where $\vec{p}$ and $v \neq 0$ are vectors in $\mathbb{R}^n$.

Equation like this is called a parametric equation for the line.

Notice here $\vec{p}$ is the pt on the line corresponding to $t=0$.

Every other pt on the line differs from $\vec{p}$ by a multiple of the vector $\vec{p}$.

Thus $\vec{p}$ is a pt on the line, $\vec{v}$ is the direction of the line.

Two equations in two unknowns.

Consider one example.

$$\begin{cases} x - y = 0 & ① \\ 3x + 4y = 9. & ② \end{cases}$$

We can use $② - 3 \times ①$, ← (this operation is called elimination.)

$$(3x+4y) - 3(x-y) = 9 - 3 0$$

$$\Rightarrow \quad 7y = 9$$

$$\Rightarrow \quad y = \frac{9}{7}.$$

Plugging in ① $\Rightarrow x = y = \frac{9}{7}$.

so the solution set is $\begin{pmatrix} \frac{9}{7} \\ \frac{9}{7} \end{pmatrix}$.

Other types of solution sets in two variables

Case 1: $\begin{cases} 3x+4y=8 \\ 6x+8y=16. \end{cases}$

the second equation is two times of the first one.

Consequently, the solution is the same as $3x+4y=8$.

so the solution is a line.

Case 2: $\begin{cases} 3x+4y=8 \\ 6x+8y=17. \end{cases}$

so there does not exist (x, y) satisfying it.

so the solution is empty.

To summarize, there are 3 possibilities for the solution set of two equations in two unknowns:

1. one pt;
2. a line;
3. no solution.

1, 3 are called degenerate cases.

Solution sets in three dimensions.

Consider $6x - y + 4z = 2$.

the solution set is a plane lying in 3-dim space.

If we solve this equation for x, we get

$$x = \frac{2 + y - 4z}{6}$$

The pts in the solution set have the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{2+y-4z}{6} \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} \frac{1}{6} \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -\frac{2}{3} \\ 0 \\ 1 \end{pmatrix}.$$

This is a parametric representation for a plane.

Any values of y and z yield a pt on the plane.

Assume $\vec{v}_1 = \begin{pmatrix} \frac{1}{6} \\ 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -\frac{2}{3} \\ 0 \\ 1 \end{pmatrix}$, $\vec{p} = \begin{pmatrix} \frac{1}{3} \\ 0 \\ 0 \end{pmatrix}$,

we can describe the solution set as

$$\{\vec{x} = \vec{p} + y\vec{v}_1 + z\vec{v}_2 \mid y, z \in \mathbb{R}\}.$$

Defn. A plane in $\mathbb{R}^n$ is a set of the form

$$\{\vec{v} = \vec{p} + y\vec{v}_1 + z\vec{v}_2 \mid y, z \in \mathbb{R}\},$$

where $\vec{p}$, $\vec{v}_1$ and $\vec{v}_2$ are vectors in $\mathbb{R}^n$ s.t $\vec{v}_1$ and $\vec{v}_2$ are not multiple of each other.

The equation above is called a parametric equation for the plane.

Two equations in 3 unknowns.

A system of two equations such as

$$\begin{cases} u - 4v + w = -2 & \leftarrow \text{one plane} \\ -2u + 10v - 3w = 4. & \leftarrow \text{another plane} \end{cases}$$

has a solution set that is intersection of 2 planes.

From geometry, we know that there are precisely 3 possibilities:

1. one line.
2. one plane.
3. no solution.

Ex. Now we use matrix notation to solve

$$\begin{cases} u - 4v + w = -2 \\ -2u + 10v - 3w = 4. \end{cases}$$

Write in matrix form,

$$\begin{pmatrix} 1 & -4 & 1 \\ -2 & 10 & -3 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}.$$

The augmented matrix is:

$$\begin{pmatrix} 1 & -4 & 1 & -2 \\ -2 & 10 & -3 & 4 \end{pmatrix}$$

When we elimination operation,

$$\begin{cases} u - 4v + w = -2 \\ 2v - w = 0. \end{cases}$$

by ② + 2 · ①.

so augmented matrix becomes

$$\begin{pmatrix} 1 & -4 & 1 & -2 \\ 0 & 2 & -1 & 0 \end{pmatrix}$$

Observe it: u, v, w are not essential in this operation, what we do is just the second row + 2 · (the first row) in the augmented matrix.

For the system of equations

$$\begin{cases} u - 4v + w = -2 \\ 2v - w = 0. \end{cases} \quad (*)$$

If we assign any value to w, then the equations allow us to solve for u and v.

Call w a $\boxed{\text{free variable}}$, and set $w = t$, to remind us that w can be any number.

Then we solve the last equation in $(*)$ for v as a fcn of $w = t$.

$$v = \frac{w}{2} = \frac{t}{2}.$$

Next, we solve the first equation for u,

$$\begin{aligned} u &= -2 + 4v - w \\ &= -2 + 4\cdot\left(\tfrac{t}{2}\right) - t \\ &= -2 + t. \end{aligned}$$

Thus, for any value of t, the vector

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -2+t \\ \frac{t}{2} \\ t \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ \frac{1}{2} \\ 1 \end{pmatrix}.$$ is a solution,

and furthermore, every solution is of the form.

So the solution set is a line through $\begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$ with vector $\begin{pmatrix} 1 \\ \frac{1}{2} \\ 1 \end{pmatrix}$.

Rmk: So you see, when we get $\begin{pmatrix} * & * & * \\ 0 & * & * \end{pmatrix}$ this form. we can go through the back-solving process.

<u>Three equation in three unknowns.</u>

Ex.
$$\begin{cases} x + + 3z = -2, \\ -3x + 2y - 5z = 2 \\ 2x - 4y + z = 1. \end{cases}$$

The augmented matrix for the system is

$$\begin{pmatrix} 1 & 0 & 3 & -2 \\ -3 & 2 & -5 & 2 \\ 2 & -4 & 1 & 1 \end{pmatrix}$$

$\Downarrow$ by elimination.

$$\begin{aligned} x + \quad + 3z &= -2 \\ 2y + 4z &= -4 \\ -4y - 5z &= 5. \end{aligned} \qquad \begin{pmatrix} 1 & 0 & 3 & -2 \\ 0 & 2y & 4z & -4 \\ 0 & -4y & -5z & 5 \end{pmatrix}$$

$\Downarrow$ by elimination

$$\begin{aligned} x + \quad + 3z &= -2 \\ 2y + 4z &= -4 \\ 3z &= -3 \end{aligned} \qquad \begin{pmatrix} 1 & 0 & 3 & -2 \\ 0 & 2 & 4 & -4 \\ 0 & 0 & 3 & -3 \end{pmatrix}$$

You see now we get $\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{pmatrix}$ this form.

We're read to back-solve.

the last one $3z = -3 \Rightarrow z = -1.$

$\Rightarrow 2y + 4(-1) = -4 \Rightarrow y = 0$

$x + 3(-1) = -2 \Rightarrow x = 1$

so the only solution to the system is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$

Solving Systems of Equations.

Row echelon form of a matrix — the goal of elimination.

Defn. The pivot of a row vector in a matrix is the first nonzero elt of that row.

Ex.
$$\begin{pmatrix} \boxed{2} & 3 & 4 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & \boxed{1} \end{pmatrix}.$$

Ex.
$$\begin{pmatrix} 0 & \boxed{2} & 4 & 2 \\ \boxed{1} & 0 & -2 & -1 \\ \boxed{-2} & 2 & 8 & 4 \end{pmatrix}$$

Ex
$$\begin{pmatrix} \boxed{2} \\ 0 \\ \boxed{1} \end{pmatrix}$$

Defn. A matrix is in row echlon form (or just echelon form) if, in each row that contains a pivot, the pivot lies to the right of the pivot in the preceding row. Any rows that contain only zeros must be at the bottom of the matrix.

like.

$$\begin{pmatrix} P & * & * & * & * & * & * & * \\ 0 & P & * & * & * & * & * & * \\ 0 & 0 & 0 & P & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & P & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & P & * \end{pmatrix}.$$

where P refers to pivots, so P must be nonzero zero numbers.

Rmk: Matrices in echelon form are set up for solution by the process of back-solving, so the goal of elimination is to put the augmented matrix into row echlon form.

Row operations and elimination.

How to make matrices into echelon form.

Ex.

$$\begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 2 & 4 & 2 \\ -2 & 2 & 8 & 4 \end{pmatrix} \xrightarrow{③+2①} \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 4 & 2 \end{pmatrix}$$

$$\xrightarrow{③-②} \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, our solution is

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3s-4t-8 \\ -4t-3s-27 \\ 3s \\ 14+t \\ t \end{pmatrix} = \begin{pmatrix} -8 \\ -27 \\ 0 \\ 14 \\ 0 \end{pmatrix} + s\begin{pmatrix} 3 \\ -3 \\ 3 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -4 \\ -4 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

where s and t are arbitrary solution.

Rmk: our solution is a parametric equation for a plane in $\mathbb{R}^5$.

To summarize, the linear system $Ax = b$ can be solved using the four steps:

1. Form the augmented matrix $M = [A, b]$.
2. Use row operations to eliminate coefficients and reduce M to row echlon form.
3. Write down the simplified system.
4. Solve the simplified system by assigning arbitrary values to the free variables and back-solving for the pivot variables.

Ex. $A = \begin{pmatrix} 2 & -3 & -2 & 2 \\ -4 & 4 & 0 & 3 \\ 8 & -8 & -1 & -7 \end{pmatrix}$ and $b = \begin{pmatrix} -4 \\ -7 \\ 13 \end{pmatrix}$.

Solve $Av = b$.

1. First write in augmented matrix.

$$\begin{pmatrix} \boxed{2} & -3 & -2 & 2 & -4 \\ \boxed{-4} & 4 & 0 & 3 & -7 \\ \boxed{8} & -8 & -1 & -7 & 13 \end{pmatrix} \leftarrow \text{not in echlon form.}$$

2.

$$\xrightarrow[\text{③}+2\text{②}]{\text{②}+2\text{①}} \begin{pmatrix} \boxed{2} & -3 & -2 & 2 & -4 \\ 0 & \boxed{-2} & -4 & 7 & -15 \\ 0 & 0 & \boxed{-1} & -1 & 1 \end{pmatrix} \leftarrow \text{in echlon form.}$$

3. The simplified system is

$$\begin{cases} 2x_1 - 3x_2 - 2x_3 + 2x_4 = -4 \\ \quad -2x_2 - 4x_3 + 7x_4 = -15 \\ \quad\quad -x_3 - x_4 = 1. \end{cases}$$

4. assign $x_4 = t$. gives $x_3 = -t-1$.

$$-2x_2 - 4x_3 + 7x_4 = -15$$

$$\Rightarrow -2x_2 - 4(-t-1) + 7t = 15$$

$$\Rightarrow 2x_2 = 15t - 11$$

$$\Rightarrow x_2 = \frac{15}{2}t - \frac{11}{2}$$

$$2x_1 - 3x_2 - 2x_3 + 2x_4 = -4$$

$$\Rightarrow 2x_1 = 3x_2 + 2x_3 - 2x_4 - 4$$

$$= 3\left(\frac{15}{2}t - \frac{11}{2}\right) + 2(-t-1) - 2t - 4$$

$$= \frac{37}{2}t - \frac{45}{2}$$

$$\Rightarrow x_1 = \frac{37}{4}t - \frac{45}{4}.$$

so the solution is

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{37}{4}t - \frac{45}{4} \\ \frac{15}{2}t - \frac{11}{2} \\ -t-1 \\ t \end{pmatrix} = \begin{pmatrix} -\frac{45}{4} \\ -\frac{11}{2} \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} \frac{37}{4} \\ \frac{15}{2} \\ -1 \\ 0 \end{pmatrix}.$$

which is a line in $\mathbb{R}^4$.