Lecture VI

Easier method to solve equation system.

Ex. $\begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & -2 \\ 2 & 0 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

1st : Augmented matrix is

$$\begin{pmatrix} 2 & 1 & -1 & 0 \\ 1 & -1 & -2 & 1 \\ 2 & 0 & -3 & 0 \end{pmatrix}. \quad \swarrow \text{ not in row echlon form.}$$

2st. $\xrightarrow[\text{exchange}]{① \text{ and } ②} \begin{pmatrix} 1 & -1 & -2 & 1 \\ 2 & 1 & -1 & 0 \\ 2 & 0 & -3 & 0 \end{pmatrix} \xrightarrow[③-2①]{②-2①} \begin{pmatrix} 1 & -1 & -2 & 1 \\ 0 & 3 & 3 & -2 \\ 0 & 2 & 1 & -2 \end{pmatrix}$

$\longrightarrow \begin{pmatrix} 1 & -1 & -2 & 1 \\ 0 & 3 & 3 & -2 \\ 0 & 0 & 1 & -\frac{2}{3} \end{pmatrix}$.

3 st.

(Old method)

so we need back-solving.

Write the simplified equation.

$$\begin{cases} x_1 - x_2 - 2x_3 = 1 \\ \quad 3x_2 + 3x_3 = -2 \\ \qquad\quad x_3 = -\frac{2}{3}. \end{cases}$$

$\Rightarrow x_3 = -\frac{2}{3}$. $\Rightarrow 3x_2 - 2 = -2$

$x_2 = 0$. $\Rightarrow x_1 - 0 - 2\cdot(-\frac{2}{3}) = 1$.

$x_1 = -\frac{1}{3}$.

(New method).

$$\begin{pmatrix} 1 & -1 & -2 & 1 \\ 0 & 3 & 3 & -2 \\ 0 & 0 & 1 & -\frac{2}{3} \end{pmatrix}.$$

$$\xrightarrow[\text{②}-3\text{③}]{\text{①}+2\text{③}} \begin{pmatrix} 1 & -1 & 0 & -\frac{1}{3} \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} \end{pmatrix}$$

$$\xrightarrow{\text{②}\times\frac{1}{3}} \begin{pmatrix} 1 & -1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} \end{pmatrix}$$

$$\xrightarrow{\text{①}+\text{②}} \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} \end{pmatrix}$$

so see write the simplified equation get

$$\begin{cases} x_1 = -\frac{1}{3} \\ x_2 = 0 \\ x_3 = -\frac{2}{3}. \end{cases}$$

Exactly solutions.

Summary.

To solve linear equation system $Ax = b$.

we can try to

make $[A, b]$ not only in row echlon form,

even in $[I, b']$ form.

$\downarrow$

imediately gives us solution.

(No need back-solving again).

Ex. $\begin{pmatrix} 5 & -2 & -5 \\ -3 & 0 & 3 \\ 0 & -3 & 0 \end{pmatrix} x = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$.

1st write the augmented matrix

$\begin{pmatrix} -5 & -2 & -5 & 2 \\ -3 & 0 & 3 & 0 \\ 0 & -3 & 0 & 3 \end{pmatrix} \leftarrow$ not in echlon form.

2nd. Change it into row echlon form.

$$\xrightarrow{②-\frac{3}{5}①}\begin{pmatrix} -5 & -2 & -5 & 2 \\ 0 & \frac{6}{5} & 6 & -\frac{6}{5} \\ 0 & -3 & 0 & 3 \end{pmatrix}$$

$$\xrightarrow{③+\frac{5}{2}②}\begin{pmatrix} -5 & -2 & -5 & 2 \\ 0 & \frac{6}{5} & 6 & -\frac{6}{5} \\ & 0 & 15 & 0 \end{pmatrix}$$

3rd. (Old method)

Write in simplied system

$$\begin{cases} -5X_1-2X_2-5X_3=2 \\ \frac{6}{5}X_2+6X_3=-\frac{6}{5} \\ 15X_3=0 \end{cases}$$

$\Rightarrow X_3=0$

$\Rightarrow X_2=-1$

$\Rightarrow 5X_1=0 \Rightarrow X_1=0.$

(New method)

$$\xrightarrow[②\times\frac{5}{6}]{③\times\frac{1}{15}}\begin{pmatrix} -5 & -2 & -5 & 2 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow[①+5③]{②-5③}\begin{pmatrix} -5 & -2 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{①+2②}\begin{pmatrix} -5 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{①\times-\frac{1}{5}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\Rightarrow X_1=0,\ X_3=0,\ X_2=-1.$

To summarize, we have 3 allowed operations on the rows of a matrix (called row operators):

R1: Add a multiple of one row to a different row.

R2: Interchange 2 rows.

R3: Multiply a row by a nonzero number.

Thm. With operations R1, R2 and R3, any matrix can be transformed into row echlon form.

Ex. Find the general solution to the system

$$\begin{cases} x_2 + 3x_3 + 2x_4 + 2x_5 = 1 \\ x_1 + 2x_2 + 3x_3 + 5x_4 + 7x_5 = 8 \\ 2x_1 + 4x_2 + 6x_3 + 9x_4 + 15x_5 = 2. \end{cases}$$

Write this as a matrix equation

$$\begin{pmatrix} 0 & 1 & 3 & 2 & 2 \\ 1 & 2 & 3 & 5 & 7 \\ 2 & 4 & 6 & 9 & 15 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ 2 \end{pmatrix}$$

The augmented matrix is

$$\begin{pmatrix} 0 & \boxed{1} & 3 & 2 & 2 & 1 \\ \boxed{1} & 2 & 3 & 5 & 7 & 8 \\ \boxed{2} & 4 & 6 & 9 & 15 & 2 \end{pmatrix}.$$

so it's not in row echlon form.

$$\begin{pmatrix} 0 & 1 & 3 & 2 & 2 & 1 \\ 1 & 2 & 3 & 5 & 7 & 8 \\ 2 & 4 & 6 & 9 & 15 & 2 \end{pmatrix} \xrightarrow[\text{① and ②}]{\text{interchange}} \begin{pmatrix} 1 & 2 & 3 & 5 & 7 & 8 \\ 0 & 1 & 3 & 2 & 2 & 1 \\ 2 & 4 & 6 & 9 & 15 & 2 \end{pmatrix}$$

$$\xrightarrow{③-2①} \begin{pmatrix} \boxed{1} & 2 & 3 & 5 & 7 & 8 \\ 0 & \boxed{1} & 3 & 2 & 2 & 1 \\ 0 & 0 & 0 & \boxed{-1} & 1 & -14 \end{pmatrix}$$ ← now in echlon form, ready for back-solving.

Hence we get

$$x_1 + 2x_2 + 3x_3 + 5x_4 + 7x_5 = 8$$
$$x_2 + 3x_3 + 2x_4 + 2x_5 = 1$$
$$-x_4 + x_5 = -14.$$

Assign $x_3 = s$, $x_5 = t$.

$\Rightarrow$ $x_4 = t + 14$.

$x_2 + 3s + 2(t+14) + 2t = 1$.

$\Rightarrow$ $x_2 = -4t - 3s - 27$.

and $x_1 = 8 - 2x_2 - 3x_3 - 5x_4 - 7x_5$

$= 8 - 2(-4t - 3s - 27) - 3s - 5(t+14) - 7t$

$= -8 + 3s - 4t$

Homogeneous and Inhomogeneous Systems.

Defn. A homogeneous system is of the form $Ax=0$.

An inhomogeneous system is of the form $Ax=b\neq 0$.

Ex. Consider the system $\begin{pmatrix} -3 & -2 & 4 \\ 15 & 8 & -18 \\ 6 & 2 & -14 \end{pmatrix} x = 0$.

Find solution.

1st. Augmented matrix.

$$\begin{array}{c} \left(\begin{array}{cccc} \boxed{-3} & -2 & 4 & 0 \\ \boxed{15} & 8 & -18 & 0 \\ \boxed{6} & 2 & -14 & 0 \end{array}\right) \end{array} \begin{array}{c} ① \\ ② \\ ③ \end{array}$$

← not in row echlon form.

2nd.

$$\xrightarrow[③+2①]{②+5①} \left(\begin{array}{cccc} \boxed{-3} & -2 & 4 & 0 \\ 0 & \boxed{-2} & 2 & 0 \\ 0 & \boxed{-2} & -6 & 0 \end{array}\right) \begin{array}{c} ① \\ ② \\ ③ \end{array}$$

$$\xrightarrow{③-②} \left(\begin{array}{cccc} \boxed{-3} & -2 & 4 & 0 \\ & \boxed{-2} & 2 & 0 \\ & & \boxed{-8} & 0 \end{array}\right).$$

← in row echlon form.

3rd. Solution of $\begin{cases} -3x_1 - 2x_2 + 4x_3 = 0 \\ \quad\ \ -2x_2 + 2x_3 = 0 \\ \qquad\qquad -8x_3 = 0. \end{cases}$

$\Rightarrow x_1 = x_2 = x_3 = 0$.

so if row echlon form is nonzero at every row, then $Ax=0$ only has zero solution.

Ex. Consider $\begin{pmatrix} -3 & -2 & 4 \\ 15 & 8 & -18 \\ 6 & 2 & -6 \end{pmatrix} X = 0.$

Find solution.

1st. The Augmented matrix is

$$\begin{pmatrix} \boxed{-3} & -2 & 4 & 0 \\ \boxed{15} & 8 & -18 & 0 \\ \boxed{6} & 2 & -6 & 0 \end{pmatrix} \leftarrow \text{not in row echlon form.}$$

2nd.

$$\xrightarrow[③+2①]{②+5①} \begin{pmatrix} \boxed{-3} & -2 & 4 & 0 \\ 0 & \boxed{-2} & 2 & 0 \\ 0 & \boxed{-2} & 2 & 0 \end{pmatrix}$$

$$\xrightarrow{③-②} \begin{pmatrix} \boxed{-3} & -2 & 4 & 0 \\ 0 & \boxed{-2} & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \leftarrow \text{in row echlon form.}$$

3rd. Solve $\begin{cases} -3x_1 - 2x_2 + 4x_3 = 0 \\ \quad -2x_2 + 2x_3 = 0. \end{cases}$

Give a free variable, Replace $x_3 = t$.

$\Rightarrow x_2 = t$.

$\Rightarrow x_1 = \frac{2}{3}t$

so $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{3}t \\ t \\ t \end{pmatrix} = t\begin{pmatrix} \frac{2}{3} \\ 1 \\ 1 \end{pmatrix}$.

Observe, our row echlon form here has a free variable place.

Comparing these 2 examples. $\Rightarrow$

Prop. The homogeneous system $Ax = 0$ has a nontrivial solution iff when change A into row echlon form, there is at least one free variable place.

Matrices in row echlon form	Free variables.

Ex. $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 8 \end{pmatrix}$.

Ex. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$

Ex. $\begin{pmatrix} 7 & 8 & 9 & 10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Ex $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

$\Rightarrow$ Cor. Any homogeneous linear system with fewer equations than unknowns has a nontrivial solution.

Structure of the solution set to an inhomogeneous system.

First state a similar thm as before.

Thm. Suppose $\vec{p}$ is a particular solution to the inhomogeneous system $Ax=b$. Then the solution set for $Ax=b$ has the form $x=\vec{p}+\vec{v}$ where $A\vec{v}=0$.

Pf: Firstly since $\vec{p}$ is solution to $Ax=b \Rightarrow A\vec{p}=b$.

$\vec{v}$ is solution to $Ax=0 \Rightarrow A\vec{v}=0$

$\Rightarrow A(\vec{p}+\vec{v})=b$.

hence $\vec{p}+\vec{v}$ is solution to $Ax=b$.

Secondly If x is a solution to $Ax=b \Rightarrow AX=b$

and $\vec{p}$ is a solution to $Ax=b \Rightarrow A\vec{p}=b$

$\Rightarrow A(x-\vec{p})=0. \Rightarrow x-\vec{p}$ is some solution $\vec{v}$ to $A\vec{v}=0$

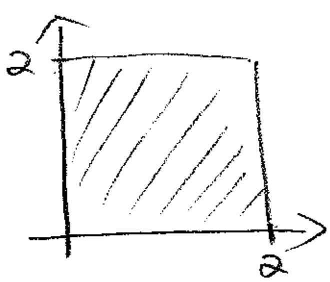
hence $x-\vec{p}=\vec{v} \Rightarrow x=\vec{p}+\vec{v}$

Square Matrix.

There is an important difference between square matrix and others.

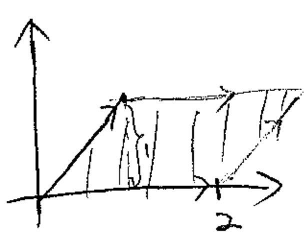
For square matrix, we can define determinant and trace, eigenvalues .....

Ex. If the parallegram I is spanned by 2 vectors $(2,0)^T$ and $(0,2)^T$.



Area (P1) $= 2 \cdot 2 = 4$.

Now how about the parallelogram which is spanned by $(1,1)^T$ and $(2,0)^T$.

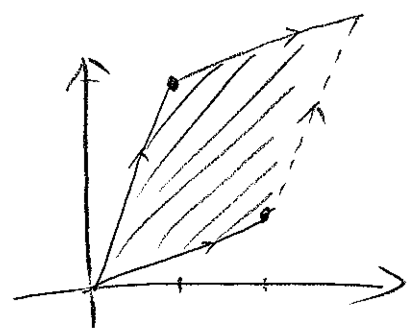


Area (P2) = ?

If you remember.

$\text{Area}(P_2) = 2 \cdot h = 2 \cdot 1 = 2$.

Next one. how about the one P_3 spanned by $(2, 1)^T$ and $(1, 3)^T$.



$\text{Area}(P_3) = ?$

We introduce the determint of $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = 2 \cdot 3 - 1 \cdot 1$.

$\text{Area}(P_3)$

$= \left| \det \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \right| \triangleq |2 \cdot 3 - 1 \cdot 1| = 5$.

You can check:

$\text{Area}(P_2) = \left| \det \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \right| = |1 \cdot 0 - 1 \cdot 2| = 2$.

$\text{Area}(P_1) = \left| \det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right| = |2 \times 2 - 0 \cdot 0| = 4$.

So determinant of 2×2 matrices gives area of the two column vectors.

Q1: How to define determinant of 3×3 matrix?

Q2: What's the geometric meaning for determinant of 3×3 matrix?

Think $\det\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} = 2\cdot2-1\cdot1 = 4-1=3.$

A1:

Ex. $\det\begin{pmatrix}2 & 3 & 5\\ 0 & 0 & 1\\ 4 & 8 & 9\end{pmatrix}$

$= 2\cdot\det\begin{pmatrix}0 & 1\\ 8 & 9\end{pmatrix} - 3\cdot\det\begin{pmatrix}0 & 1\\ 4 & 9\end{pmatrix} + 5\cdot\det\begin{pmatrix}0 & 0\\ 4 & 8\end{pmatrix}$

$= 2\cdot(0-8) - 3(0-4) + 5(0-0)$

$= -16+12+0$

$= -4.$

Ex. $\det\begin{pmatrix}-3 & 2 & 8\\ 0 & 4 & 0\\ 3 & 2 & 1\end{pmatrix}$

$= -3\cdot\det\begin{pmatrix}4 & 0\\ 2 & 1\end{pmatrix} - 2\cdot\det\begin{pmatrix}0 & 0\\ 3 & 1\end{pmatrix} + 8\cdot\det\begin{pmatrix}0 & 4\\ 3 & 2\end{pmatrix}$

$= -3\cdot(4-0) - 2\cdot(0-0) + 8(0-12)$

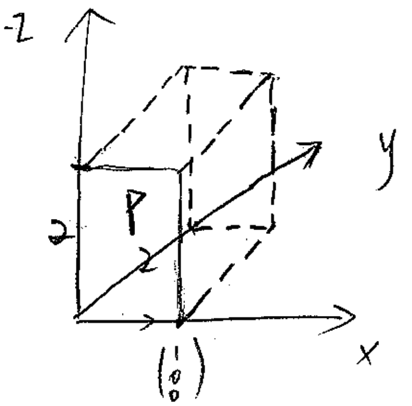
$= -12+96$

$= 84.$

A2: Can you guess?

Yes, volume.

Consider $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.



$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$

generate the parallelepiped P.

$$\mathrm{Vol}(P) = \left|\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right|.$$

$\mathrm{Vol}(P) = 1 \times 2 \times 2 = 4.$

$$\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 1 \cdot \det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - 0 \cdot \det \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} + 0 \cdot \det \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$= 1 \cdot 4 = 4. \quad \checkmark.$$

Now. How to define determinant of a 4x4 matrix?

Ex. $\det \begin{pmatrix} 0 & 2 & 3 & 4 \\ 2 & 0 & 3 & 1 \\ 1 & 1 & 1 & 3 \\ 0 & 4 & -3 & 0 \end{pmatrix}.$

Can you first guess?

Yes. $\triangleq 0\cdot\det\begin{pmatrix}0 & 3 & 1\\ 1 & 1 & 3\\ 4 & -3 & 0\end{pmatrix} - 2\cdot\det\begin{pmatrix}2 & 3 & 1\\ 1 & 1 & 3\\ 0 & -3 & 0\end{pmatrix}$

$+3\cdot\det\begin{pmatrix}2 & 0 & 1\\ 1 & 1 & 3\\ 0 & 4 & 0\end{pmatrix} - 4\cdot\det\begin{pmatrix}2 & 0 & 3\\ 1 & 1 & 1\\ 0 & 4 & -3\end{pmatrix}$

And $\det\begin{pmatrix}2 & 3 & 1\\ 1 & 1 & 3\\ 0 & -3 & 0\end{pmatrix} = 2\cdot\det\begin{pmatrix}1 & 3\\ -3 & 0\end{pmatrix} - 3\cdot\det\begin{pmatrix}1 & 3\\ 0 & 0\end{pmatrix}$

$+1\cdot\det\begin{pmatrix}1 & 1\\ 0 & -3\end{pmatrix}$

$= 2\cdot(0+9) - 3\cdot(0+0) + 1\cdot(-3-0)$

$= 18 - 0 - 3 = 15.$

$\det\begin{pmatrix}2 & 0 & 1\\ 1 & 1 & 3\\ 0 & 4 & 0\end{pmatrix} = 2\cdot\det\begin{pmatrix}1 & 3\\ 4 & 0\end{pmatrix} - 0\cdot\det\begin{pmatrix}1 & 3\\ 0 & 0\end{pmatrix}$

$+1\cdot\det\begin{pmatrix}1 & 1\\ 0 & 4\end{pmatrix}$

$= 2\cdot(0-12) - 0 + 1\cdot(4-0)$

$= -24+4 = -20$

$\det\begin{pmatrix}2 & 0 & 3\\ 1 & 1 & 1\\ 0 & 4 & -3\end{pmatrix} = 2\cdot\det\begin{pmatrix}1 & 1\\ 4 & -3\end{pmatrix} - 0\cdot\det\begin{pmatrix}1 & 1\\ 0 & 3\end{pmatrix}$

$+ 3\det\begin{pmatrix}1 & 1\\ 0 & 4\end{pmatrix}$

$$= 2\cdot(3-4) - 0 + 3\cdot(4-0)$$
$$= -14 + 12 = -2.$$

Now $\doteq$ $-2\cdot 15 + 3\cdot(-20) - 4\cdot(-2)$
$$= -82.$$

Very tedious calculation!

Can we make our calculation easier? Yes.

Like. we just want to make zeros more.

Ex. $\det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 2 & 4 & 0 \\ -1 & 2 & 3 & 4 \end{pmatrix}.$

$$= 1\cdot\det\begin{pmatrix} 3 & 0 & 0 \\ 2 & 4 & 0 \\ 2 & 3 & 4 \end{pmatrix}$$

$$= 1\cdot 3\cdot\det\begin{pmatrix} 4 & 0 \\ 3 & 4 \end{pmatrix}$$

$$= \underbrace{1\cdot 3\cdot 4\cdot 4}. = 48.$$

What's this? Just diagonal entries' product!

Fact : $\det \begin{pmatrix} \lambda_1 & & & \\ * & \lambda_2 & & \\ * & * & \ddots & \\ * & * & \cdots & \lambda_n \end{pmatrix} \overset{?}{=} \lambda_1 \lambda_2 \cdots \lambda_n$!

lower triangular matrix

Then how about $\det \begin{pmatrix} \lambda_1 & * & * & \cdots & * \\ & \lambda_2 & & & \vdots \\ & & \ddots & & * \\ & & & & \lambda_n \end{pmatrix} \overset{?}{=} \lambda_1 \lambda_2 \cdots \lambda_n$.

because we have a fact :

Prop. Let A be $n \times n$ matrix, then

$$\det(A^T) = \det(A).$$

Ex. $A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$, $A^T = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$.

$\det(A) = 2 - 12 = -10$, $\det(A^T) = 2 - 12 = -10$. ✓

So you see : If we can make our matrix A become A' in row echlon form, we can just use the products of diagonal entries to give determinant of matrix A' in row echlon form.

How does $\det(A')$ help us to know $\det(A)$.

i.e. What's the relation of $\det(A)$ and $\det(A')$.

Prop. Let A be an $n\times n$ matrix.

1. If the matrix B is obtained from A by adding a multiple of one column (rows) to another, then $\det(B) = \det(A)$.

2. If the matrix B is obtained from A by interchanging two columns (rows), then $\det(B) = -\det(A)$.

3. If the matrix B is obtained from A by multiplying a column (row) by a constant c, then $\det(B) = c\det(A)$.

We see these statements from examples.

1. $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & 1 \end{pmatrix}$.

then $\det(A) = \det(B)$!

2. $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 0 & 2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 3 \\ 2 & 1 & 1 \end{pmatrix}$

$D = \begin{pmatrix} 2 & 0 & 0 \\ -5 & -2 & 0 \\ 2 & 1 & 1 \end{pmatrix}$, so $\det(A) = -\det(B) = -(-\det(C)) = \det(D) = -4$.

Ex.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 7 & 11 & 12 \\ 7 & 8 & 9 & 10 & 100 & 1 \\ 2 & 3 & 4 & 5 & 6 & 7 \\ 10 & 10 & 10 & 10 & 10 & 10 \\ 0 & 1 & 0 & 8 & 12 & 13 \end{pmatrix}. \quad \det = ?$$

Yes. ① and ⑤ are the same.

$$\det(A) = \det \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & & & & & \cdots \\ & & & & & \end{pmatrix}$$

$$= 0 \, !$$

Fact: If a matrix A has 2 columns (rows) are multiples of each other.

$\Longrightarrow \det(A) = 0$.

After investigating the special classes.
Now we give the general strategy to calculate determinant.

Do by examples.

Ex.
$$\det \begin{pmatrix} 0 & 2 & 3 & 4 \\ 2 & 0 & 3 & 1 \\ 1 & 1 & 1 & 3 \\ 0 & 4 & -3 & 0 \end{pmatrix}$$

. make it into row echlon form.

$$\xrightarrow{①,\ ③\ \text{exchange}} \underbrace{= -\det}_{} \begin{pmatrix} \boxed{1} & 1 & 1 & 3 \\ \boxed{2} & 0 & 3 & 1 \\ 0 & \boxed{2} & 3 & 4 \\ 0 & \boxed{4} & -3 & 0 \end{pmatrix}$$

(remember the change of determinate)

$$\xrightarrow{② - 2①} = -\det \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -2 & 1 & -5 \\ 0 & 2 & 3 & 4 \\ 0 & 4 & -3 & 0 \end{pmatrix}$$

$$\xrightarrow{③ + ②} = -\det \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -2 & 1 & -5 \\ 0 & 0 & 4 & -1 \\ 0 & 4 & -3 & 0 \end{pmatrix}$$

$$\xrightarrow{④ + 2②} = -\det \begin{pmatrix} \boxed{1} & 1 & 1 & 3 \\ 0 & \boxed{-2} & 1 & -5 \\ 0 & 0 & \boxed{4} & -1 \\ 0 & 0 & \boxed{-1} & -10 \end{pmatrix}$$

$$\xrightarrow{④ + \frac{1}{4}③} = -\det \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -2 & 1 & -5 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & -10\frac{1}{4} \end{pmatrix}$$

$$= -1\cdot(-2)\cdot 4\cdot\left(-10\tfrac{1}{4}\right)$$

$$= -82.$$

Ex. $\det\begin{pmatrix} -1 & -9 & 10 \\ -7 & -19 & 26 \\ -2 & -10 & 12 \end{pmatrix}$

Make it into row echlon form.

$$\xrightarrow{②-7①} = \det\begin{pmatrix} \boxed{-1} & -9 & 10 \\ 0 & \boxed{44} & -44 \\ \boxed{-2} & -10 & 12 \end{pmatrix}$$

$$\xrightarrow{③-2①} = \det\begin{pmatrix} -1 & -9 & 10 \\ 0 & 44 & -44 \\ 0 & 8 & -8 \end{pmatrix}$$

$$\xrightarrow{②\times\frac{1}{44}} = 44\det\begin{pmatrix} -1 & -9 & 10 \\ 0 & 1 & -1 \\ 0 & 8 & -8 \end{pmatrix}$$

$$\xrightarrow{③-8②} = 44\det\begin{pmatrix} -1 & -9 & 10 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= 44\cdot(-1)\cdot 1\cdot(0) = 0.$$

More properties of the determinant.

Prop. Suppose A and B are $n \times n$ matrices. then $\det(AB) = \det(A) \cdot \det(B)$.

Rmk: ① If $\det A = 0$, $\Rightarrow \forall$ $n \times n$ matrices B,

$$\det(AB) = 0.$$

② We've told that $AB \neq BA$ in general.

but we still have

$$\det(AB) = \det(BA).$$

Defn. An $n \times n$ matrix A is $\boxed{\text{invertible}}$ if there is an $n \times n$ matrix B s.t $AB = I$ and $BA = I$.

A matrix B with this property is called $\boxed{\text{an inverse}}$ of A, written in A^{-1}.

Rmk: ③ $\det(AB) = \det(I) = 1$

$\Rightarrow \det A \cdot \det B = 1$.

Tells us:

(1) A is invertible $\Leftrightarrow \det A \neq 0$.

and $\det(A^{-1}) = 1/\det A$.

(2) A is not invertible $\Leftrightarrow \det A = 0$.

It's important to know how to calculate

A^{-1}, for an invertible A.

Recall. If A is invertible,

we can make row operations

$$\Longrightarrow \begin{pmatrix} \boxed{*} & * & * & * \\ & \boxed{*} & * & * \\ & & \boxed{*} & * \\ & & & \boxed{*} \end{pmatrix}$$

continue.
all are nonzero.

$$\Longrightarrow \begin{pmatrix} \boxed{*} & & & 0 \\ & \boxed{*} & & \\ & & \ddots \ \boxed{*} & \\ 0 & & & \boxed{*} \end{pmatrix}$$

$$\xrightarrow{\text{by saling}} \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ 0 & & & & 1 \end{pmatrix}$$

And in fact, doing row operations $\underset{\text{as}}{\overset{\text{same}}{=}}$ multiply a specific matrix on the left.

In the end.

$$\begin{array}{c} A \\ \text{by row operations} \downarrow \\ I \end{array} \iff \begin{array}{c} \text{the whole process is} \\ \downarrow \text{by multiply } A^{-1} \text{ on the left.} \\ (A^{-1})A = I. \end{array}$$

So how can we see this A^{-1}?

There is a trick.

We do row operations not only on A.

but on $[A, I]$.

After enough row operations (i.e, multiply A^{-1} on the left)

$$\Rightarrow [A, I] \longrightarrow [I, A^{-1}].$$

So in the end, the righ-side matrix is our A^{-1}.

Ex.
$$A = \begin{pmatrix} 0 & 0 & -3 & -5 \\ 1 & -2 & -6 & -7 \\ -1 & 3 & 7 & 7 \\ 0 & -1 & -3 & -3 \end{pmatrix}.$$

Consider $[A, I]$.

$$[A, I] = \begin{pmatrix} 0 & 0 & -3 & -5 & 1 & 0 & 0 & 0 \\ 1 & -2 & -6 & -7 & 0 & 1 & 0 & 0 \\ -1 & 3 & 7 & 7 & 0 & 0 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[\text{exchange.}]{1,\ 2} \begin{pmatrix} 1 & -2 & -6 & -7 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & -5 & 1 & 0 & 0 & 0 \\ -1 & 3 & 7 & 7 & 0 & 0 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{3 + 1} \begin{pmatrix} \boxed{1} & -2 & -6 & -7 & 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{-3} & -5 & 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & \boxed{-1} & -3 & -3 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[\text{exchange}]{2,\ 3} \begin{pmatrix} 1 & -2 & -6 & -7 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -3 & -5 & 1 & 0 & 0 & 0 \\ 0 & -1 & -3 & -3 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{4 + 2} \begin{pmatrix} 1 & -2 & -6 & -7 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -3 & -5 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & -3 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{③ \times -\frac{1}{3}} \begin{pmatrix} 1 & -2 & -6 & -7 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{5}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & -2 & -3 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{④ + 2③} \begin{pmatrix} 1 & -2 & -6 & -7 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{5}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & -\frac{2}{3} & 1 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{④ \times 3} \begin{pmatrix} 1 & -2 & -6 & -7 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{5}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 3 & 3 & 3 \end{pmatrix}$$

$$\xrightarrow[① + 7④]{③ - \frac{5}{3}④} \begin{pmatrix} 1 & -2 & -6 & 0 & -14 & 22 & 21 & 21 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 3 & -5 & -5 & -5 \\ 0 & 0 & 0 & 1 & -2 & 3 & 3 & 3 \end{pmatrix}$$

$$\xrightarrow[① + 6③]{② - ③} \begin{pmatrix} 1 & -2 & 0 & 0 & 4 & -8 & -9 & -9 \\ 0 & 1 & 0 & 0 & -3 & 6 & 6 & 5 \\ 0 & 0 & 1 & 0 & 3 & -5 & -5 & -5 \\ 0 & 0 & 0 & 1 & -2 & 3 & 3 & 3 \end{pmatrix}$$

$$\xrightarrow{① + 2②} \begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 4 & 3 & 1 \\ 0 & 1 & 0 & 0 & -3 & 6 & 6 & 5 \\ 0 & 0 & 1 & 0 & 3 & -5 & -5 & -5 \\ 0 & 0 & 0 & 1 & -2 & 3 & 3 & 3 \end{pmatrix}$$

so $A^{-1} = \begin{pmatrix} -2 & 4 & 3 & 1 \\ -3 & 6 & 6 & 5 \\ 3 & -5 & -5 & -5 \\ -2 & 3 & 3 & 3 \end{pmatrix}$