Lecture VII.

After introduce invertible matrices, now we talk about the concepts "singular" and "nonsingular".

Here I talk very differently from the book.

We call invertible matrices <u>nonsingular</u>

Otherwise <u>singular</u>. (only for $n \times n$ matrices)

Prop. For nonsingular matrix A, we can solve

$$Ax = b \quad \text{for arbitrary vector } b.$$

and the solution is unique.

Pf: Since A is nonsingular $\Rightarrow$ A is invertible

$\Rightarrow A^{-1}$ exists.

so $$A^{-1}(Ax) = A^{-1}b.$$

$$\Rightarrow Ix = A^{-1}b$$

$$\Rightarrow x = A^{-1}b.$$

so $x = A^{-1}b$ is the unique solution.

Prop. For singular Matrice A, $\exists\, b$ s.t

$Ax = b$ has no solution.

Pf: Suppose $Ax = b$ alway has solution for arbitrary b.

then $\exists\, \vec{x}_1$ s.t $A\vec{x}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\exists\, \vec{x}_2$ s.t $A\vec{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

......

$\exists\, \vec{x}_n$ s.t $A\vec{x}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$.

Put them together.

$$(A\vec{x}_1, A\vec{x}_2, \cdots, A\vec{x}_n) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = I.$$

$$\parallel$$

$$A\underbrace{(\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n)}_{B} = I.$$

so $\exists\, B$ s.t $AB = I$. $\Rightarrow \det A \cdot \det B = \det(I) = 1$.

$\Rightarrow \det A \neq 0$.

Contradiction.

Prop. If A is an $n \times n$ matrix, then the homogeneous equation $Ax = 0$ has a nonzero solution iff the matrix is singular.

Pf: If the matrix is nonsingular. A has inverse A^{-1}!

$$Ax = 0 \implies A^{-1}(Ax) = 0$$

$$\implies Ix = x = 0.$$

so $x = 0$ is the unique solution.

If the matrix is singular.

When we solve $Ax = 0$.

the row echelon form. $\begin{pmatrix} \square & & & \\ & \square & & \\ & & \ddots \ \square & \\ & & & 0 \end{pmatrix} \leftarrow$ free variable place.

hence we have a free variable.

so the solution won't be zero only.

Rmk: For $Ax = 0$, if there is one solution, then there are many solutions.

Cor. A nonsingular $\Longleftrightarrow$ $Null(A) = \{0\}$ $\Leftrightarrow$ $\det A \neq 0$

A singular $\Longleftrightarrow$ $Null(A)$ is nontrivial

$\Leftrightarrow$ $\det(A) = 0$

Ex. For $\begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & 3 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$,

suppose we already know the solution set is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$$

hence can we tell me the solution set of the associated homogeneous system.

$$\begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & 3 \end{pmatrix} x = 0.$$

Yes, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$.

Rmk: For any 2 solutions for above homogeneous system.

like $\vec{x} = t_1 \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$, $\vec{y} = t_2 \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$.

we can easily see $\vec{x} + \vec{y} = (t_1 + t_2) \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$ is also a solution.

$\forall\ a \in \mathbb{R}$, $a\vec{x} = (at_1) \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$ is also a solution.

And a set having these 2 properties is very special.

From your homework, you can see

$S = \{ \begin{pmatrix} t \\ 0 \end{pmatrix} \mid t > 0 \}$ can never be a solution set.

Now we continue to discuss the solution set of homo system.

Nullspaces.

Defn. The nullspace of a matrix A is set of all solutions to the homogeneous system of linear equations $Ax = 0$. The nullspace of A is denoted by $null(A)$.

Restate the properties in remark.

Prop. Let A be an $n \times m$ matrix.

1. Suppose x and y are vectors in $null(A)$, then
$x + y$ is also in $null(A)$.
2. Suppose x is in $null(A)$ and a is a number.
then ax is also in $null(A)$.

Next, if a is a number, then

$$ax = ab_1x_1 + ab_2x_2 + \cdots + ab_kx_k$$

is also a combination of $x_1, \cdots, x_k$.

Hence ax belongs to $\mathrm{span}(x_1, \cdots, x_k)$. $\square$

Subspaces

Defn. A nonempty subset V of $\mathbb{R}^n$ that has the following 2 properties is called a subspace of $\mathbb{R}^n$.

1. If x and y are vectors in V, then $x+y$ is also in V.
2. If x is in V, a is a number, then ax is also in V.

Cor. Any linear combination of vectors in V is also in V.

Rmk: 0 is always an element of V.

$\forall x \in V$, $0 \cdot x = 0 \in V$.

And $\{0\}$ is a subspace.

the total space $\mathbb{R}^n$ is also a subspace.

These two subspaces will be refered to as the trivial subspaces.

We've discovered that the nullspace of a matrix consists of the set of all linear combinations of a few vectors.

In this class, we study this structure in more detail.

The span of a set of vectors.

Defn. If $x_1, x_2, \cdots, x_k$ are vectors in $\mathbb{R}^n$, we define the span of $x_1, \cdots, x_k$ to be the set of all linear combinations of $x_1, \cdots, x_k$, and we will denote this set by

$$\text{span}\{x_1, \cdots, x_k\}.$$

Prop. Suppose that $x_1, \cdots, x_k$ are vectors in $\mathbb{R}^n$, and set

$$V = \text{span}(x_1, \cdots, x_k).$$

1. If x and y are vectors in V, then $x+y$ is also in V.
2. If x is in V and a is a number, then ax is also in V.

Pf: Suppose $x, y \in V$, hence $\exists$ $b_1, \cdots, b_k$, $c_1, \cdots, c_k$ s.t

$$x = b_1 x_1 + b_2 x_2 + \cdots + b_k x_k.$$

$$y = c_1 x_1 + c_2 x_2 + \cdots + c_k x_k.$$

Consequently,

$$x + y = (b_1 + c_1) x_1 + (b_2 + c_2) x_2 + \cdots + (b_k + c_k) x_k,$$

is a linear combination of $x_1, \cdots, x_k$, hence in V.

In fact, $\text{span}\{x_1, \cdots, x_k\}$ is a subspace.

Q: Given $V = \text{span}\{x_1, \cdots, x_k\}$, how to determine if $x \in V$?.

Let us us guess:

If $x = a_1 x_1 + \cdots + a_k x_k$.

$$\Rightarrow \quad \underbrace{(x_1 \cdots x_k)}_{\text{a matrix}} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix} = \underset{\text{a vector}}{x}. \qquad (*)$$

so if $x \in V$, that means we have solution for $(*)$.

if $x \notin V$, " " we don't have solution for $(*)$.

So the strategy to determine if $x \in V = \text{span}(x_1, \cdots, x_k)$ is:

1. Form the matrix $X = [x_1, x_2, \cdots, x_k]$.
2. Solve the system $X\vec{a} = \vec{x}$.

   (a) If there are no solutions, $\vec{x}$ is not in $\text{span}(x_1, \cdots, x_k)$.

   (b) If $\vec{a} = (a_1, a_2, \cdots, a_k)^T$ is a solution, then

   $$\vec{x} = a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_k\vec{x}_k$$

   is in $\text{span}(\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_k)$.

(see: we even know the precise combination of $\vec{x}$ by $\vec{x}_1, \cdots, \vec{x}_k$.)

Ex. Consider $v_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, and $v_4 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

Show that $\text{span}(v_1, v_2) = \mathbb{R}^2$.

$\text{span}(v_1, v_2, v_4) = \mathbb{R}^2$.
$\text{span}(v_2, v_3) = \mathbb{R} \cdot v_2$.

If we can show $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \text{span}(v_1, v_2)$.

and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \text{span}(v_1, v_2)$.

Since $\begin{pmatrix} u \\ v \end{pmatrix} = u\begin{pmatrix} 1 \\ 0 \end{pmatrix} + v\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. that is enough to show

$\text{span}(v_1, v_2) = \mathbb{R}^2$.

So First whether $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \text{span}(v_1, v_2)$.

1. $\begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}$.

2. $\begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

the augmented matrix is $\begin{pmatrix} -1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ ← not in echlon form

$\longrightarrow \begin{pmatrix} \boxed{-1} & 1 & 1 \\ 0 & \boxed{3} & 2 \end{pmatrix}$ ← in echlon form.

Write simplified system as $\begin{cases} -a_1 + a_2 = 1 \\ 3a_2 = 2 \end{cases}$

$\Rightarrow a_2 = \frac{2}{3}, \quad a_1 = -\frac{1}{3}.$ ✓.

Second whether $\begin{pmatrix}0\\1\end{pmatrix} \in span(v_1, v_2)$.

1. $\begin{pmatrix}-1 & 1\\2 & 1\end{pmatrix}$.

2. $\begin{pmatrix}-1 & 1\\2 & 1\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$

the augmented matrix is $\left(\begin{array}{cc|c}-1 & 1 & 0\\2 & 1 & 1\end{array}\right)$ ← not in echlon form

$\longrightarrow \left(\begin{array}{cc|c}-1 & 1 & 0\\0 & 3 & 1\end{array}\right)$ ← in echlon form.

Write simplified system as $\begin{cases}-a_1 + a_2 = 0\\ 3a_2 = 1\end{cases}$

$\Rightarrow a_2 = \frac{1}{3},\ a_1 = \frac{1}{3}.$ ✓.

So $Span(v_1, v_2) = \mathbb{R}^2$.

Since $span(v_1, v_2)$ is already the whole space. so $\exists\ a_1, a_2$ s.t $v_4 = a_1 v_1 + a_2 v_4$.

so $span(v_1, v_2, v_4) = \mathbb{R}^2$.

linear dependence and independence.

Ex. $v_4 = v_2 - v_1 \iff v_1 - v_2 + v_4 = 0.$

say v_4 has relation with v_2, v_1. / $\exists$ a nontrivial combination of v_2, v_2 and v_4 $= 0$.

Defn. The vectors $x_1, x_2, \cdots, x_k$ are linearly independent if the only linear combination of them that is equal to the zero vector is trivial one. Otherwise, call them linearly dependent.

i.e.

If $\exists$ $c_1, c_2, \cdots, c_k$ s.t

$$\vec{0} = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_k\vec{x}_k .$$

then $c_1 = c_2 = \cdots = c_k = 0$

Ex. For 2 vectors v_1, v_2,

If v_1, v_2 are linearly dependent,

then $a v_1 + b v_2 = 0$. with $a \neq 0$ or $b \neq 0$,

If $a \neq 0$, $\Rightarrow v_1 = -\frac{b}{a} v_2$.

If $b \neq 0$, $\Rightarrow v_2 = -\frac{a}{b} v_1$.

Anyway, linearly dependent for 2 vectors v_1, v_2

$\iff$ One of them is a multiple of the other.

If you remember, yesterday we said

For a square matrix A, if $\exists$ 2 rows (columns), one of them is a multiple of the other, then

$\det A = 0$.

So here we can restate the fact:

For a square matrix A, if $\exists$ 2 row (column) vectors are linearly dependent, then $\det A = 0$.

To generalise, how about $\exists$ 3 row (column) vectors are linearly dependent, $\Longrightarrow \det A = 0$?

Yes! even to k.

Fact: $\exists$ k row (column) vectors of A are linearly dependent. $\Longleftrightarrow \det A = 0$.

Pf (by example) If $A = [v_1, v_2, v_3]$.

And v_1, v_2, v_3 are linearly dependent.

$\Rightarrow$ may be $v_1 + v_2 - v_3 = 0$.

$\Rightarrow v_3 = v_1 + v_2$.

So $\det A = \det[\overset{①}{v_1}, \overset{②}{v_2}, \overset{③}{v_3}] \xRightarrow[=]{③-①} \det[v_1, v_2, v_2]$

$$\xrightarrow{③ - ②} = \det[v_1, v_2, 0] = 0. \qquad \square$$

Ex. Consider the following vectors:

$$v_1 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix}.$$

whether they are linearly independent.

If linearly dependent, find a combination that is equal to 0.

Just assume $x_1 \vec{v_1} + x_2 \vec{v_2} + x_3 \vec{v_3} = 0$.

to see whether x_1, x_2, x_3 all need be 0.

$\Longleftrightarrow$ $[v_1, v_2, v_3]x = 0$ has nontrivial solution or not.

$\uparrow$ becomes solving equation.

$$\begin{pmatrix} -1 & -2 & -2 \\ -2 & -1 & -3 \\ 2 & 2 & 4 \end{pmatrix} x = 0$$

1st the augmented matrix is

$$\left(\begin{array}{ccc|c} \boxed{-1} & -2 & -2 & 0 \\ \boxed{-2} & -1 & -3 & 0 \\ \boxed{2} & 2 & 4 & 0 \end{array}\right)$$

2nd $\xrightarrow[③+2①]{②-2①} \begin{pmatrix} \boxed{-1} & -2 & -2 & 0 \\ 0 & \boxed{3} & 1 & 0 \\ 0 & \boxed{-2} & 0 & 0 \end{pmatrix} \xrightarrow{③+\frac{2}{3}②} \begin{pmatrix} \boxed{-1} & -2 & -2 & 0 \\ 0 & \boxed{3} & 1 & 0 \\ 0 & 0 & \boxed{\frac{2}{3}} & 0 \end{pmatrix}$

3rd. No free variables. $\Longrightarrow x_1 = x_2 = x_3 = 0$. $\Longrightarrow$ v_1, v_2, v_3 are linearly independent.

Bases of a subspace.

"A minimal spanning set" is called a basis.

Defn. A set of vectors $\{x_1, x_2, \cdots, x_k\}$ in a subspace V is a basis for V if it has the following properties:

1. The vectors span V;
2. The vectors are linearly independent.

Prop. Suppose that V is a subspace of $\mathbb{R}^n$.

1. V has a basis;
2. Any 2 bases for V have the same number of elements. (this number is called dimension of V.).

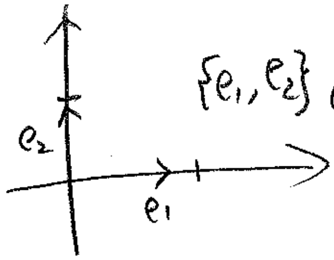
Rmk: (1) In $\mathbb{R}^n$, there is a standard basis $\{e_1, \cdots, e_n\}$, where $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\cdots$, $e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$.

You can easy see $e_1, \cdots, e_n$ are linearly independent. (why?).

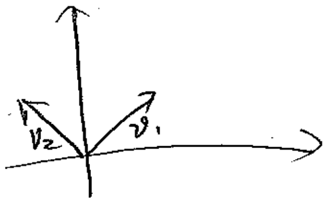
And any $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

$= x_1 e_1 + x_2 e_2 + \cdots + x_n e_n \in \text{span}\{e_1, \cdots, e_n\}$.

(2) Bases are not unique.



$\{e_1, e_2\}$ are basis.



$\{v_1, v_2\}$ are a basis, too.

There arise 2 questions to basis.

Q1: We know a null space of A is a subspace.
how to find a basis for a nullspace?

Q2: Given vectors $v_1, v_2, \cdots, v_k$, how to find a basis for span$\{v_1, v_2, \cdots, v_k\}$.?

(Both answers are not unique.)

Answer of Q1: (by example)

Ex. Find a basis for Null(A),
where $A = \begin{pmatrix} 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & -1 \end{pmatrix}$

In fact, you'll see, the basis comes out when we're solving Null(A).

To solve Null(A).

$$Ax = 0.$$

1st. The augmented matrix is

$$\begin{pmatrix} \boxed{1} & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & \boxed{1} & 2 & -1 & 0 \end{pmatrix} \leftarrow$$ in row echlon form.

$$\begin{array}{cccccc} & \uparrow & & \uparrow & \uparrow & \end{array}$$

free variable.

2nd. Replace $x_2 = r$, $x_4 = s$, $x_5 = t$.

so $x_3 + 2x_4 - x_5 = 0$

$\Rightarrow x_3 = x_5 - 2x_4 = t - 2s$.

$x_1 - x_2 + 2x_4 = 0$

$\Rightarrow x_1 = -2x_4 + x_2 = r - 2s$.

so $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} r-2s \\ \boxed{r} \\ t-2s \\ \boxed{s} \\ \boxed{t} \end{pmatrix} = r\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -2 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

So let $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} -2 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ solution set $= \mathrm{Null}(A) = \mathrm{span}\{v_1, v_2, v_3\}$.

3rd. Now need to prove $\{v_1, v_2, v_3\}$ are linearly independent.

(Here is the standard argument to prove $\{v_1, v_2, v_3\}$ are linearly independent.)

Suppose $\exists\ c_1, c_2, c_3$ s.t $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$.

hence $\Rightarrow \begin{pmatrix} c_1 - 2c_2 \\ c_1 \\ -2c_2 + c_3 \\ c_2 \\ c_3 \end{pmatrix} = 0$.

$\Rightarrow c_1 = c_2 = c_3 = 0$.

$\Rightarrow \{v_1, v_2, v_3\}$ are linearly independent.

So we can see $\{v_1, v_2, v_3\}$ is one basis for $Null(A)$.

so dim of $Null(A) = 3$.

Observation: $\dim(Null(A)) = \#$ of free variables!

Is this just by lucky?

Not really, this is true always.

We can see why.

There are special places from free variables for each vector,

So there is no way from other vectors' combination to get one vector.

Answer of Q2:

This is also a standard method. (by example)

Ex. Find a basis for $V = \text{span}\{v_1, v_2, v_3\}$, where

$$v_1 = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix}.$$

Step 1: Start from v_1, we include v_1 into our basis.

Step 2: Check whether $v_2 \in \text{span}\{v_1\}$.

i.e. whether v_2 is a multiple of v_1.

No.

If $a\begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} \Rightarrow 0 \cdot a = -2$. No.

So include v_2 into our basis also.

Step 3: Check whether $v_3 \in \text{span}\{v_1, v_2\}$.

(if No, include v_3 into our basis also, and then our basis is $\{v_1, v_2, v_3\}$;
if yes, then our basis is just $\{v_1, v_2\}$.)

How to check $v_3 \in \text{span}\{v_1, v_2\}$.

$\Leftrightarrow$ Whether $\exists\, a, b$ s.t. $v_3 = a v_1 + b v_2$.

$\Leftrightarrow$ $[v_1, v_2]\begin{pmatrix} a \\ b \end{pmatrix} = v_3$.

$\Longleftrightarrow$ $[v_1, v_2]\,x = v_3$ has solution or not?

becomes solving equation system again.

$$\begin{pmatrix} 0 & -2 \\ -2 & -1 \\ 2 & 2 \end{pmatrix} x = \begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix}.$$

1st. Augmented matrix.

$$\begin{pmatrix} 0 & \boxed{-2} & -2 \\ \boxed{-2} & -1 & -3 \\ \boxed{2} & 2 & 4 \end{pmatrix} \leftarrow \text{not in row echlon form.}$$

2nd. $\xrightarrow[\text{exchange}]{①,②} \begin{pmatrix} -2 & -1 & -3 \\ 0 & -2 & -2 \\ 2 & 2 & 4 \end{pmatrix}$

$\xrightarrow{③+①} \begin{pmatrix} -2 & -1 & -3 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{pmatrix}$

$\xrightarrow{③+\frac{1}{2}②} \begin{pmatrix} -2 & -1 & -3 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{pmatrix}$

so $\begin{cases} -2x_1 + (-1)x_2 = -3 \\ \quad -2x_2 = -2 \end{cases}$

$\Rightarrow x_2 = 1,\ x_1 = -1.$

so we have solution.

$\Rightarrow v_3 = -v_1 + v_2 \Rightarrow v_3 \in \operatorname{span}\{v_1, v_2\}.$

One basis is $\{v_1, v_2\}$.

We are ready to move on to systems of differential equations.

Motivation: The SIR model of an epidemic.

The use of systems greatly expands our ability to model applications. Let's start by modding an epidemic.

First we assume the following facts about the disease:

- The disease is of short duration and rarely fatal.
- The disease spreads through contact between individuals.
- Individuals who have recovered from the disease are immune.

These features are present in measles, the mumps, and in the common cold.

The listed features allow us to construct a model.

Divide the population into 3 groups:

- The susceptible, $S(t)$.

are those individuals who have never had the disease.

- The infected, $I(t)$.

are those who are currently ill with the disease.

• The recovered, $R(t)$.
are those who have had the disease and are now immune.

The total population is the sum of these three,

$$N = S + I + R.$$

Now we need compute the rate of change for each of these.

① $\dfrac{dN}{dt} = 0.$

Since the disease is of short duration and rarely fatal, we may ignore deaths and births. This means the total population N is constant.

② $\dfrac{dS}{dt}.$

Since the disease spreads through contacts between susceptible and infected individuals, the rate of change is proportional to the number of contacts.

So can assume $\dfrac{dS}{dt} = -aSI.$

where a is a postive constant.

Putting everything together, we get the system of equations

$$\begin{cases} S' = -aSI, \\ I' = aSI - bI, \\ R' = bI. \end{cases}$$

This is the $\boxed{\text{SIR model}}$.

If we write $u_1 = S$, $u_2 = I$, $u_3 = R$. $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$.

we get $\underbrace{(\vec{u})' = f(\vec{u})}_{\text{vector vertion.}}$, where $f(\vec{u}) = \begin{pmatrix} -au_1u_2 \\ au_1u_2 - bu_2 \\ bu_2 \end{pmatrix}$

Initial Value Problem. for SIR system

The intial value Problem for SIR system with initial time t_0

is $\begin{cases} S' = -aSI, \\ I' = aSI - bI, \\ R' = bI, \end{cases}$ with $\begin{matrix} S(t_0) = S_0, \\ I(t_0) = I_0, \\ R(t_0) = R_0. \end{matrix}$

Using vector notation,

$$\vec{u}' = f(\vec{u}), \quad \text{with } \vec{u}(t_0) = \begin{pmatrix} S_0 \\ I_0 \\ R_0 \end{pmatrix}.$$

③ $\frac{dI}{dt}$.

$I(t)$ change in 2 ways.

First, it increases as susceptible individuals get sick.
we've already determined that rate to be aSI.

In addition, infected individuals get well and then pass into the recovered population.

Assuming there is a fairly standard time in which a recovery takes place, the rate of recovery is propotional to the number of infected.

So there is a positive constant b st the rate of recoveries is bI.

Putting together: $\frac{dI}{dt} = aSI - bI$.

④ $\frac{dR}{dt}$.

We've determined that $\frac{dR}{dt} = bI$.

General 1st-order system

Defn. If the unknown fcns are $x_1(t), x_2(t), \cdots, x_n(t)$, then the general first-order system has the form

$$x_1' = f_1(t, x_1, x_2, \cdots, x_n)$$
$$x_2' = f_2(t, x_1, x_2, \cdots, x_n)$$
$$\cdots\cdots$$
$$x_n' = f_n(t, x_1, x_2, \cdots, x_n).$$

where $f_1, f_2, \cdots,$ and f_n are fcns of the $n+1$ variables $t, x_1, x_2, \cdots,$ and x_n.

Defn. A solution to the system would be an n-tuple of fcns $x_1(t), x_2(t), \cdots,$ and $x_n(t)$ that satisfies the equations of the system.

Rmk: We'll always require that the number of equations is equal to the number of unknowns, and this number is called the dimension of the system.

Defn. A system of dimension 2 is called a planar system.

The SIR model has dimension 3.

In fact, if we don't take R in, since $R = N - S - I$, then $\begin{cases} S' = -aSI, \\ I' = aSI - bI \end{cases}$ can also represents the SIR model.

Now, it's also a planar system.

We claim know enough of general first-order systems is sufficient to solve higher-order systems!!!

In fact, we can always reduct higher-order systems into first-order systems.

<u>Reduction of higher-order systems to first-order system.</u>

Ex. Find a first-order system equivalent to the third-order, nonlinear equation

$$x''' + x \cdot x'' = cost.$$

The basic idea is to introduce new variables.

We introduce $u_1 = x$, $u_2 = x'$, $u_3 = u''$,

$$\text{so} \quad \begin{cases} u_1' = u_2 \\ u_2' = u_3 \\ u_3' + x u_3 = cost \end{cases}$$

$$\Longleftrightarrow \quad \begin{cases} u_1' = u_2 \\ u_2' = u_3 \\ u_3' = cost - x u_3. \end{cases}$$

Now if we solve the 1st-order system,

then u_1 is our desired solution x for

$$x''' + x \cdot x'' = \cos t.$$

Ex. $w^{(4)} + w^{(3)} + ww' = \sin t$. reduction to 1st-order system.

We introduce $u_1 = w$, $u_2 = w'$, $u_3 = w''$, $u_4 = w'''$.

So
$$\begin{cases} u_1' = u_2 \\ u_2' = u_3 \\ u_3' = u_4 \\ u_4' + u_4 + u_1 u_2 = \sin t \end{cases}$$

$$\Leftrightarrow \begin{cases} u_1' = u_2 \\ u_2' = u_3 \\ u_3' = u_4 \\ u_4' = \sin t - u_4 - u_1 u_2. \end{cases}$$