Lecture VIII

Linear Systems with Constant Coefficients.

We concentrate on the system

$$\vec{y}' = A\vec{y}$$

where A is a matrix with constant entries.

Called linear system with constant coefficients.

Ex. $\begin{cases} x' = 2y + 3x \\ y' = 3x + z \\ z' = z + x. \end{cases}$ ✓

Ex. $\begin{cases} S' = -aSI \\ I' = aSI - bI \\ R' = bI \end{cases}$ ✗

Q: How to solve this system?

Recall, if there is one equation as

$$x' = ax.$$

then it's obvious that $x = Ce^{at}$. C is a constant

but now we're dealing with

$$\vec{y}' = A\vec{y}.$$

In analogy to the solution $x = Ce^{at}$, let's look for solutions of the form

$$\vec{y}(t) = e^{\lambda t}\vec{v}$$, where $\vec{v}$ is a vector with constants for entries.

The entries of $\vec{v}$ and λ are yet unknown and can be chosen to satisfy our equation system.

If we plug in $\vec{y}(t) = e^{\lambda t}\vec{v}$ to $\vec{y}' = A\vec{y}$.

we get $(e^{\lambda t}\vec{v})' = A \cdot e^{\lambda t}\vec{v}$.

$$\Rightarrow \cancel{e^{\lambda t}} \cdot \lambda\vec{v} = A \cdot \cancel{e^{\lambda t}}\vec{v}$$

$$\Rightarrow A\vec{v} = \lambda\vec{v}.$$

So we need our $\lambda, \vec{v}$ satisfy

$$\boxed{A\vec{v} = \lambda\vec{v}}.$$

they have special name.

Defn. Suppose A is an $n \times n$ matrix. A number λ is called $\boxed{\text{an eigenvalue}}$ of A if there is a <u>nonzero</u> vector $\vec{v}$ s.t $A\vec{v} = \lambda\vec{v}$. (*) Any vector $\vec{v}$ satisfying (*) is called $\boxed{\text{an eigenvector}}$ associated with the eigenvalue λ.

Observation:

1) $\vec{v} \neq 0$ is necessary;

2) If $\vec{v}$ is an eigenvector associated to λ,

then so is $k\vec{v}$, $\forall k \in \mathbb{R}$.

(because $A(k\vec{v}) = k \cdot A\vec{v} = k \cdot (\lambda \vec{v}) = \lambda (k\vec{v})$.)

In fact, we will show that the set of eigenvectors associated to a particular eigenvalue is a subspace of $\mathbb{R}^n$.

Let's state our result formally.

Thm. Suppose λ is an eigenvalue of the matrix A and $\vec{v}$ is an associated eigenvector.

Then $\vec{x}(t) = e^{\lambda t}\vec{v}$ is a solution to the system $\vec{x}' = A\vec{x}$ and satisfies the initial condition $\vec{x}(0) = \vec{v}$.

So we can see, our task to solve $\vec{x}' = A\vec{x}$ becomes to find eigenvalues and eigenvectors of A.

Let's first discuss techniques for computing them.

First, let's rewrite $A\vec{v} = \lambda\vec{v}$ as

$$A\vec{v} - \lambda\vec{v} = 0$$

$$\Leftrightarrow \quad \underbrace{(A - \lambda I)}\,\vec{v} = 0$$

$$\Leftrightarrow \det(A - \lambda I) = 0.$$

Since $\vec{v}$ is nonzero, we need $A - \lambda I$ is singular

so λ is the value which makes $A - \lambda I$ singular

and $\vec{v}$ is one nontrivial solution of $(A - \lambda I)\vec{v} = 0$.

(i.e. $0 \neq \vec{v} \in \text{Null}(A - \lambda I)$.)

First we give a name to $\det(A - \lambda I)$, in fact, to $(-1)^n \det(A - \lambda I)$.

Defn. If A is $n\times n$ matrix, the polynomial

$$p(\lambda) = (-1)^n \det(A - \lambda I) = \det(\lambda I - A)$$

is called $\boxed{\text{the characteristic polynomial}}$ of A,

and the equation $p(\lambda) = (-1)^n \det(\lambda I - A) = 0$

is called $\boxed{\text{the characteristic equation}}$.

Return to our restriction to eigenvalue λ, which is $\det(A - \lambda I) = 0$.

Prop. The eigenvalues of A are the roots of its characteristic polynomial.

Ex. Find the eigenvalues of the matrix

$$A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}.$$

The characteristic polynomial is

$$p(\lambda) = \det(\lambda I - A)$$

$$= \det\left(\begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} - \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}\right)$$

$$= \det\left(\begin{pmatrix} \lambda+4 & -6 \\ 3 & \lambda-5 \end{pmatrix}\right)$$

$$= (\lambda+4)(\lambda-5) + 18$$

$$= \lambda^2 - \lambda - 2$$

$$= (\lambda-2)(\lambda+1).$$

hence the eigenvalues of A are 2 and -1.

Ex. Find the eigenvalues of the matrix.

$$A = \begin{pmatrix} 3 & 6 & 7 \\ & 4 & 8 \\ & & 5 \end{pmatrix}$$

The characteristic polynomial is

$$p(\lambda) = \det(\lambda I - A)$$

$$= \det\left(\begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix} - \begin{pmatrix} 3 & 6 & 7 \\ & 4 & 8 \\ & & 5 \end{pmatrix}\right)$$

$$= \det\left(\begin{pmatrix} \lambda-3 & -6 & -7 \\ & \lambda-4 & -8 \\ & & \lambda-5 \end{pmatrix}\right)$$

$$= (\lambda-3)(\lambda-4)(\lambda-5)$$

hence the eigenvalues of A are 3, 4 and 5.

So you may know immediately the eigenvalues of

$$\begin{pmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix}$$ is $\lambda_1, \lambda_2, \cdots,$ and λ_n.

<u>Finding eigenvectors.</u>

Let's see the eigenvectors associated with eigenvalue λ are simply the vectors in $\mathrm{Null}(A-\lambda I)$.

Defn. The subspace consisting of all eigenvectors for a given eigenvalue λ is called $\boxed{\text{the eigenspace of } \lambda}$.

Let's state formally.

Prop. Let A be an $n \times n$ matrix, and let λ be an eigenvalue of A. The set of all eigenvectors associated with λ is equal to the nullspace of $A - \lambda I$.

Hence, the eigenspace of λ is a subspace of $\mathbb{R}^n$.

Ex. For $A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$.

Find the eigenvectors for A.

For eigenvalue 2, the eigenspace is $\text{Null}(A-2I)$

$$= \text{Null}\left(\begin{pmatrix} -6 & 6 \\ -3 & 3 \end{pmatrix}\right)$$

Solve $\begin{pmatrix} -6 & 6 \\ -3 & 3 \end{pmatrix}\vec{v} = 0$

$\Rightarrow \vec{v}_1 = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For eigenvalue -1, the eigenspace is $\text{Null}(A+I)$

$$= \text{Null}\left(\begin{pmatrix} -3 & 6 \\ -3 & 6 \end{pmatrix}\right).$$

Solve $\begin{pmatrix} -3 & 6 \\ -3 & 6 \end{pmatrix}\vec{v} = 0$

$\Rightarrow \vec{v}_2 = t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Now we use this example to our original problem,

solving $\vec{x}' = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}\vec{x}$.

We've said $e^{\lambda t}\vec{v}$ is a solution to above system, where λ is eigenvalue of A and $\vec{v}$ is eigenvector associated with λ.

So we know

$\vec{X}_1(t) = e^{2t}\begin{pmatrix}1\\1\end{pmatrix}$ and $\vec{X}_2(t) = e^{-t}\begin{pmatrix}2\\1\end{pmatrix}$ are 2 solutions

to $\quad X' = \begin{pmatrix}-4 & 6\\-3 & 5\end{pmatrix}\vec{x}$.

Q: How could we know all solutions of

$$X' = \begin{pmatrix}-4 & 6\\-3 & 5\end{pmatrix}\vec{x} \text{ ?}$$

Can you guess?

A: All solutions to $X' = \begin{pmatrix}-4 & 6\\-3 & 5\end{pmatrix}\vec{x}$ are

of the form $\quad c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$, $\quad c_1, c_2$ are constants.

Properties of Linear Systems.

Thm. Suppose $\vec{x}_1$ and $\vec{x}_2$ are solutions to the homogeneous linear system. $\vec{x}' = A\vec{x}$. (*)

If c_1 and c_2 are any constants, then

$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$ is also a solution to (*).

Pf: Just plug in $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$ into (*).

$$\text{Left side} = (c_1\vec{x}_1 + c_2\vec{x}_2)' = c_1 \cdot \vec{x}_1' + c_2\vec{x}_2'$$

$$= c_1 A\vec{x}_1 + c_2 A\vec{x}_2.$$

$$= A(c_1\vec{x}_1 + c_2\vec{x}_2).$$

$$= \text{Right side}. \quad \square.$$

Rmk: This thm only works for linear and homogeneous system.

In fact, we can generalise the thm.

Thm. Suppose that $\vec{x}_1, \vec{x}_2, \cdots,$ and $\vec{x}_k$ are all solutions to the homogeneous linear system $\vec{x}' = A\vec{x}$. Then any linear combination of $\vec{x}_1, \vec{x}_2, \cdots$ and $\vec{x}_k$ is also a solution.

i.e. $\forall c_1, c_2, \cdots, c_k \in \mathbb{R}$,

$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_k\vec{x}_k(t)$ is a solution to $\vec{x}' = A\vec{x}$.

In our example,

hence $\vec{x}(t) = c_1 e^{2t}\begin{pmatrix}1\\1\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}2\\1\end{pmatrix}$ is also a solution.

Converse Key Question:

Can all solutions to a linear homogeneous system be expressed as a linear combination of certain special solutions?

(In our example, same question: Is any solution of $\vec{x}' = \begin{pmatrix}-4 & 6\\ -3 & 5\end{pmatrix}\vec{x}$ of the form $c_1 e^{2t}\begin{pmatrix}1\\1\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}2\\1\end{pmatrix}$ for some c_1, c_2?

To answer this question, we first need explain more concepts.

Linear independence and dependence (for fcns).

Prop. Suppose that $\vec{x}_1(t)$, $\vec{x}_2(t)$, $\cdots$ and $\vec{x}_k(t)$ are solutions to the n-dim system $\vec{x}' = A\vec{x}$ defined on the interval $I = (\alpha, \beta)$.

1. If the vectors $\vec{x}_1(t_0)$, $\vec{x}_2(t_0)$, $\cdots$, and $\vec{x}_k(t_0)$ are linearly independent for some $t_0 \in I$, then there are constants $c_1, c_2, \cdots$, and c_k, not all zero, such that

$c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_k\vec{x}_k(t) = 0$ for all $t \in I$.

In particular, $\vec{x}_1(t)$, $\vec{x}_2(t)$, $\cdots$, and $\vec{x}_k(t)$ are linearly dependent for all $t \in I$.

2. If for some $t_0 \in I$, the vectors $\vec{x}_1(t_0)$, $\vec{x}_2(t_0)$, $\cdots$, and $\vec{x}_k(t_0)$ are linearly independent, then $\vec{x}_1(t)$, $\vec{x}_2(t)$, $\cdots$, and $\vec{x}_k(t)$ are linearly independent for all $t \in I$.

So we can make the following definition.

Defn. A set of k solutions to the linear system $\vec{x}' = A\vec{x}$ is linearly independent if it is linearly independent for any one value of t.

And we know from proposition that if a set of solutions is linearly independent, then it is linearly independent for all values of t.

Now we're ready to answer the question.

Thm. Suppose $\vec{x}_1$, $\cdots$, and $\vec{x}_n$ are linearly independent solutions to the n-dim linear system $\vec{x}'(t) = A\vec{x}(t)$.
Then any solution $\vec{x}(t)$ can be expressed as a linear combination of $\vec{x}_1$, $\cdots$, and $\vec{x}_n$. That is, there are constants c_1, $\cdots$, and c_n s.t.

$$\vec{x}(t) = c_1\vec{x}_1(t) + \cdots + c_n\vec{x}_n(t) \text{ for all } t.$$

Defn. A set of n linearly independent solutions to a homogeneous linear system of dim n is $\boxed{\text{a fundamental set}}$ of solutions

Solution Strategy

For $\vec{x}' = A\vec{x}$.

Find n linearly independent solutions $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n$.

The general solution is the set of all linear combinations

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_n\vec{x}_n(t).$$

Observation: So the key step is two show that n solutions are linearly independent.

According to previous result, we only have to show this for one value of t. One way to do this is to use determinants, we need show that for one value of t,

$$W(t) = \det([\vec{x}_1(t), \vec{x}_2(t), \cdots, \vec{x}_n(t)]) \neq 0.$$

The fcn $W(t)$ is called the Wronskian of $\vec{x}_1(t), \vec{x}_2(t), \cdots, \vec{x}_n(t)$.

Go back to our example.

$$W(t) = \det\left(\left[e^{2t}\begin{pmatrix}1\\1\end{pmatrix}, e^{-t}\begin{pmatrix}2\\1\end{pmatrix}\right]\right)$$

$$= \begin{pmatrix} e^{2t} & 2e^{-t} \\ e^{2t} & e^{-t} \end{pmatrix}$$

$$= e^{t} - 2e^{t}$$

$$= -e^{t} \neq 0 \quad \text{for } t=1.$$

so $\vec{x}_1(t) = e^{2t}\begin{pmatrix}1\\1\end{pmatrix}$, $\vec{x}_2(t) = e^{-t}\begin{pmatrix}2\\1\end{pmatrix}$ are linearly independent

hence they are a fundament set of solutions.

So our solutions to $\vec{x}' = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}\vec{x}$ are

of $c_1 \cdot e^{2t}\begin{pmatrix}1\\1\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}2\\1\end{pmatrix}$. $\square$

In fact, the property of "linearly independence" of solutions holds in general for the case of k distinct eigenvalues in dim n.

Thm. Suppose $\lambda_1, \lambda_2, \cdots$, and λ_k are distinct eigenvalues of an $n\times n$ matrix A. Suppose $v_i \neq 0$ is an eigenvector for λ_i.

$$= (\lambda-1)(\lambda-4)+2$$

$$= \lambda^2 - 5\lambda + 6$$

$$= (\lambda-2)(\lambda-3).$$

so eigenvalues of A are 2 and 3.

Step 2: Find eigenvectors.

For eigenvalue 2,

$$Null(A-2I) = Null\left(\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}\right)$$

$$= t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For eigenvalue 3,

$$Null(A-3I) = Null\left(\begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix}\right)$$

$$= t\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Step 3: Solutions to system are

$$c_1 e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix},\quad c_1, c_2 \text{ are constants.}$$

then the solutions $\vec{x}_1(t) = e^{\lambda_1 t}\vec{v}_1, \cdots, \vec{x}_k(t) = e^{\lambda_k t}\vec{v}_k$ are linearly independent.

If $k=n$, then the solutions

$\vec{x}_1(t) = e^{\lambda_1 t}\vec{v}_1, \cdots, \vec{x}_n(t) = e^{\lambda_n t}\vec{v}_n$ are a fundamental set of solutions.

Now we summarize everything for A as 2×2 matrix.

Thm. Suppose A is a 2×2 matrix with real eigenvalues $\lambda_1 \neq \lambda_2$. Suppose that $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors associated with the eigenvalues. Then the general solution to the system $\vec{x}\,' = A\vec{x}$ is

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2,$$

where c_1, c_2 are arbitrary constants.

Ex. Solve $\vec{x}\,' = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \vec{x}$.

Step 1: Find eigenvalues.

$$\det((\lambda I - A)) = \det\left(\begin{pmatrix} \lambda-1 & -2 \\ 1 & \lambda-4 \end{pmatrix}\right).$$

Conclusion: For dim 2 linear system. (Planar linear system).
The case of 2 distinct real eigenvalues is solved!

Q: How about other cases?

- One real eigenvalue with multiplicity 2.
- No real eigenvalue, 2 conjugate complex eigenvalues.

Let's solve these 2 cases also.

Complex eigenvalues.

Ex. Solve $\vec{x}' = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix} \vec{x}$.

Step 1: Find eigenvalues.

$$\det((\lambda I - A)) = \det\left(\begin{pmatrix} \lambda & -1 \\ 2 & \lambda-2 \end{pmatrix}\right)$$

$$= \lambda^2 - 2\lambda + 2.$$

So $\lambda_{1,2} = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$.

hence eigenvalues of A are $\lambda = 1+i$ and $\bar{\lambda} = 1-i$.

Step 2: Find eigenspace for $\lambda = 1+i$.

$$\mathrm{Null}((A-\lambda I)) = \mathrm{Null}(A-(1+i)I)$$

$$= \mathrm{Null}\left(\begin{pmatrix} -(1+i) & 1 \\ -2 & 2-(1+i) \end{pmatrix}\right) = \mathrm{Null}\left(\begin{pmatrix} -1-i & 1 \\ -2 & 1-i \end{pmatrix}\right)$$

We know it should be singular.

so the second row must be a multiple of the first row.

hence only need use the information of first row.

that is $(-1-i)w_1 + w_2 = 0$

$$\Rightarrow \vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = t\begin{pmatrix} 1 \\ 1+i \end{pmatrix}.$$

Now we need find eigenspace for $\bar{\lambda} = 1-i$.

(Here is the main difference with the case of distinct real eigenvalues.)

Since $A\vec{w} = \lambda\vec{w}$, after conjugation,

$$\Rightarrow \overline{A\vec{w}} = \overline{\lambda\vec{w}}$$

$$\Rightarrow A(\overline{\vec{w}}) = \bar{\lambda}(\overline{\vec{w}})$$

Complex-valued solutions are prefered in some situations, however, in other situations, it is important to find real-valued solutions. Fortunately, the real and imaginary parts of a complex solution provide the needed fondamental set of solutions.

Prop. Suppose A is an $n\times n$ matrix with real coefficients, and suppose that $z(t) = \vec{x}(t) + i\vec{y}(t)$ is a solution to the system $\vec{z}' = A\vec{z}$ (*).

(a) The complex conjugate $\overline{(\vec{z})} = \vec{x} - i\vec{y}$ is also a solution to (*).

(b) The real and imaginary parts $\vec{x}$ and $\vec{y}$ are also solutions to (*). Furthermore, if $\vec{z}$ and $\overline{(\vec{z})}$ are linearly independent, so are $\vec{x}$ and $\vec{y}$.

Pf: To prove (a), since $(\vec{z}(t))' = A(\vec{z}(t))$

$$\Rightarrow \quad (\overline{\vec{z}(t)})' = \overline{A(\vec{z}(t))}$$

$$= A\,\overline{\vec{z}(t)}.$$

$\Rightarrow \overline{\vec{z}(t)}$ is also a solution.

Hence $\overline{(\vec{w})}$ is one eigenvector for $\bar{\lambda}$.

so we don't need to find eigenspace for $\bar{\lambda}$.

just take conjugation of $\vec{w}$.

Step 3: Our solution is

$$\vec{X}(t) = c_1 e^{\lambda t} \vec{w} + c_2 e^{\bar{\lambda} t} \overline{(\vec{w})}, \quad c_1, c_2 \text{ are constants.}$$

Summarize this fact in thm.

Thm. Suppose that A is a 2×2 matrix with complex conjugate eigenvalues λ and $\bar{\lambda}$. Suppose that $\vec{w}$ is an eigenvector associated with λ. Then the general solution to the system $\vec{X}' = A\vec{X}$ is $\vec{X}(t) = c_1 e^{\lambda t} \vec{w} + c_2 e^{\bar{\lambda} t} \overline{(\vec{w})}$, c_1, c_2 are constants.

Rmk: The solutions in thm are complex valued.

We finish our example in Step 3.

Step 3': $\vec{x}(t) = c_1 e^{(1+i)t} \begin{pmatrix} 1 \\ 1+i \end{pmatrix} + c_2 e^{(1-i)t} \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$

(b). Since $\vec{x} = \frac{1}{2}(\vec{z} + \overline{\vec{z}})$

$\vec{y} = \frac{1}{2i}(\vec{z} - \overline{\vec{z}})$

$\Rightarrow \vec{x}, \vec{y}$ are linear combinations of $\vec{z}$ and $\overline{\vec{z}}$.

$\Rightarrow \vec{x}, \vec{y}$ are also solutions to $(*)$.

Ex. Find a fundamental set of real solutions for $\vec{x}' = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix} \vec{x}$.

We already get $\vec{z} = e^{(1+i)t} \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$.

$$= e^t e^{it} \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

$$= e^t \cdot \begin{pmatrix} \cos t + i\sin t \\ (\cos t + i \sin t)(1+i) \end{pmatrix}$$

$$= e^t \begin{pmatrix} \cos t + i \sin t \\ (\cos t - \sin t) + i(\cos t + \sin t) \end{pmatrix}$$

$$= e^t \begin{pmatrix} \cos t \\ \cos t - \sin t \end{pmatrix} + i e^t \begin{pmatrix} \sin t \\ \cos t + \sin t \end{pmatrix}.$$

hence a fundamental set of real solutions is

$$\left\{ \vec{x}(t) = e^t \begin{pmatrix} \cos t \\ \cos t - \sin t \end{pmatrix}, \ \vec{y}(t) = e^t \begin{pmatrix} \sin t \\ \cos t + \sin t \end{pmatrix} \right\}.$$

And all real solutions to (*) are of the form

$$c_1 \vec{x}(t) + c_2 \vec{y}(t), \quad c_1, c_2 \text{ are constants.}$$

Ex. $\vec{x}' = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} \vec{x}$.

Step 1: Find eigenvalues.

$$\det((\lambda I - A)) = \det\left(\begin{pmatrix} \lambda+1 & -2 \\ 2 & \lambda-1 \end{pmatrix}\right)$$

$$= (\lambda+1)(\lambda-1) + 4$$

$$= \lambda^2 + 3.$$

so $\lambda = \sqrt{3}i$, $\bar{\lambda} = -\sqrt{3}i$ are eigenvalues of A.

Step 2: Find eigenvectors for $\lambda = \sqrt{3}i$.

$$\text{Null}(A - \lambda I) = \text{Null}\left(\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} - \begin{pmatrix} \sqrt{3}i & \\ & \sqrt{3}i \end{pmatrix}\right)$$

$$= \text{Null}\begin{pmatrix} -1-\sqrt{3}i & 2 \\ -2 & 1-\sqrt{3}i \end{pmatrix}.$$

Remember this matrix is singular, the second row show be a multiple of the first row.

so $(-1-\sqrt{3}i)w_1 + 2w_2 = 0$

so $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1+\sqrt{3}i \end{pmatrix}$.

Step 3: $\vec{z} = e^{\lambda t}\vec{w}$

$$= e^{\sqrt{3}i \cdot t}\begin{pmatrix} 2 \\ 2+\sqrt{3}i \end{pmatrix}$$

$$= \begin{pmatrix} 2\cos\sqrt{3}t + 2i\sin\sqrt{3}t \\ (\cos\sqrt{3}t + i\sin\sqrt{3}t)(2+\sqrt{3}i) \end{pmatrix}$$

$$= \begin{pmatrix} 2\cos\sqrt{3}t + 2i\sin\sqrt{3}t \\ (2\cos\sqrt{3}t - \sqrt{3}\sin\sqrt{3}t) + i(\sqrt{3}\cos\sqrt{3}t + 2\sin\sqrt{3}t) \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 2\cos\sqrt{3}t \\ 2\cos\sqrt{3}t - \sqrt{3}\sin\sqrt{3}t \end{pmatrix}}_{\vec{x}(t)} + i\underbrace{\begin{pmatrix} 2\sin\sqrt{3}t \\ \sqrt{3}\cos\sqrt{3}t + 2\sin\sqrt{3}t \end{pmatrix}}_{\vec{y}(t)}.$$

so real solutions are of the form

$c_1\vec{x}(t) + c_2\vec{y}(t)$, c_1, c_2 are constants.