One eigenvalue with multiplicity 2.

Ex. For the case $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \vec{x}$.

Step 1: Find eigenvalues of A.

$$\det(\lambda I - A) = \det\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \det\left(\begin{pmatrix} \lambda-1 & -1 \\ 0 & \lambda-1 \end{pmatrix}\right)$$

$$= (\lambda-1)^2.$$

so $\lambda_{1,2} = 1$ are eigenvalues of A.

Step 2: Find eigenspaces of $\lambda = 1$.

$$\text{Null}(A - \lambda I) = \text{Null}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$$

$$= t\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector.

so $\vec{x}_1(t) = e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is one solution.

Q: But this is a dim 2 system, where is the other one in fundament set of solutions?

Summary.

Thm. Suppose A is a 2×2 matrix with one eigenvalue λ of multiplicity 2, and suppose that the eigenspace of λ has dimension 1. Let $\vec{v}_1$ be a nonzero eigenvector, and choose $\vec{v}_2$ s.t

$$(A-\lambda I)\vec{v}_2 = \vec{v}_1, \quad \text{then}$$

$$\vec{x}_1(t) = e^{\lambda t}\vec{v}_1 \quad \text{and} \quad \vec{x}_2(t) = e^{\lambda t}(\vec{v}_2 + t\vec{v}_1)$$

form a fundamental set of solutions to $\vec{x}' = A\vec{x}$.

Ex. $\vec{x}' = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}\vec{x}$.

Step 1: Find eigenvalues of A.

$$\det(\lambda I - A) = \det\left(\begin{pmatrix} \lambda+1 & 1 \\ -1 & \lambda+3 \end{pmatrix}\right)$$

$$= (\lambda+1)(\lambda+3)+1$$

$$= \lambda^2+4\lambda+4$$

$$= (\lambda+2)^2$$

so $\lambda=2$ is eigenvalue of A of multiplicity 2.

Step 2: Find eigenvector $\vec{v}_1$ of A associated with -2.

$$\text{Null}(A+2I) = \text{Null}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

hence $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

First guess what the next solution will be.

$$\vec{x}_2(t) = e^{\lambda t}\,(v_2 + \underset{\Delta}{t}\, v_1)$$

this is the key.

Now we need to find v_2 s.t

$\vec{x}_2(t)$ is really a solution.

Plug in $\vec{x}_2(t) = e^{\lambda t}(\vec{v}_2 + t\vec{v}_1)$

we get

Left side $\quad (\vec{x}_2(t))' = \lambda e^{\lambda t}(\vec{v}_2 + t\vec{v}_1) + e^{\lambda t}\vec{v}_1$

$$= e^{\lambda t}\left((\lambda\vec{v}_2 + \vec{v}_1) + \lambda t\vec{v}_1\right)$$

Right sider $\quad A\vec{x}_2(t) = A\cdot e^{\lambda t}(\vec{v}_2 + t\vec{v}_1)$

$$= A\cdot e^{\lambda t}\vec{v}_2 + t\cdot e^{\lambda t}\lambda\vec{v}_1$$

we need $\quad e^{\lambda t}\left((\lambda\vec{v}_2 + \vec{v}_1) + \cancel{\lambda t\vec{v}_1}\right) = A\cdot e^{\lambda t}\vec{v}_2 + \cancel{\lambda t\cdot e^{\lambda t}\vec{v}_1}$

so $\quad \cancel{e^{\lambda t}}(\lambda\vec{v}_2 + \vec{v}_1) = A\cdot\cancel{e^{\lambda t}}\vec{v}_2$

so $\quad (A - \lambda I)\vec{v}_2 = \vec{v}_1.$

Step 3: Find $\vec{v}_2$.

We need $(A+2I)\vec{v}_2 = \vec{v}_1$.

Solve $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

hence $\begin{cases} x_1 - x_2 = 1 \\ x_1 - x_2 = 1 \end{cases}$

so choose $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. (see $\vec{v}_2$ is not unique at all.)

Step 4. $\vec{x}_1(t) = e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{x}_2(t) = e^{-2t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$

form a fundamental set of solutions

to $\vec{x}' = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}\vec{x}$,

i.e. solutions are of the form

$$c_1 e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} t+1 \\ t \end{pmatrix}.$$

The plase space and the plase plane.

Suppose in general that we have a planar system

$$\begin{cases} y_1' = f(t, y_1, y_2) \\ y_2' = f(t, y_1, y_2). \end{cases}$$

If $\vec{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ is a solution, then we can look at the solution curve $t \longmapsto \vec{y}(t)$ in the plane.

Defn. The y_1y_2-plane is called the $\boxed{\text{phase plane}}$.

The solution curve is called a phase plane plot, or a solution curve in the phase plane.

Defn. $\boxed{\text{An autonomous system}}$ is a system in which the right-hand side does not depend explicitly on the independent variable.

Ex. A planar autonomous system has the form

$$\begin{cases} x' = f(x, y), \\ y' = g(x, y). \end{cases}$$

Ex. If you notice, $\vec{x}' = A\vec{x}$ are autonomous systems.

For planar autonomous systems, consider y_1y_2-plane (phase plane)

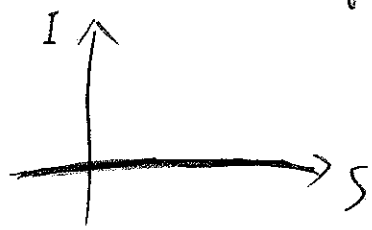
Defn. For an arbitrary autonomous system $\vec{x}' = f(\vec{x})$, a vector $\vec{x}_0$ for which the right-hand side vanishes. (i.e $f(\vec{x}_0)=0$) is called $\boxed{\text{an equilibrium pt}}$. The fcn $\vec{X}(t) = \vec{x}_0$ satisfies the equation and is called an equilibrium solution.

Ex. $\begin{cases} S' = -SI \\ I' = SI - I. \end{cases}$ where is the equilibrium pt? (S_0, I_0)

For right-side, we need $\begin{cases} -S_0 I_0 = 0 \\ S_0 I_0 - I_0 = 0 \end{cases}$

so $I_0 = 0$.

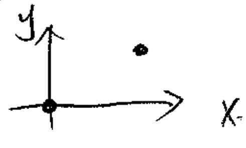
so $\{(S_0, 0)\}$ are equilibrium pts.



Ex. $\begin{cases} x' = (x-3)y \\ y' = (2-y)x. \end{cases}$ where is the equilibrium pt? (x_0, y_0)

For right-side, we need $\begin{cases} (x_0-3)y_0 = 0 \\ (2-y_0)x_0 = 0. \end{cases}$

so $(3, 2)$ and $(0,0)$ are the 2 equilibrium pts.



Phase Plane Portraits.

Now that we know how to solve linear planar systems with constant coefficients, let's find out what the solutions look like.

We'll examine the six most important cases here.

There are several more.

To set the stage, we will be considering the system

$$y' = Ay$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ and } \vec{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}.$$

The characteristic polynomial is

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{pmatrix}$$

$$= (\lambda - a_{11})(\lambda - a_{22}) - a_{12}a_{21}$$

$$= \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}$$

$$= \lambda^2 - \underbrace{(\operatorname{tr} A)}_{T}\lambda + \underbrace{\det A}_{D}.$$

$$= \lambda^2 - T\lambda + D.$$

Real eigenvalue Case:

$\boxed{T^2 - 4D > 0}$ $\lambda_1 = \frac{T + \sqrt{T^2 - 4D}}{2}$, $\lambda_2 = \frac{T - \sqrt{T^2 - 4D}}{2}$

$$\vec{y}(t) = C_1 e^{\lambda_1 t} \vec{v}_1 + C_2 e^{\lambda_2 t} \vec{v}_2 .$$

Two particular solutions are $C_1 e^{\lambda_1 t} \vec{v}_1$ and $C_2 e^{\lambda_2 t} \vec{v}_2$, the so-called $\boxed{\text{exponential solutions}}$.

$\boxed{\text{Exponential solutions}}$

$$\vec{y}(t) = C e^{\lambda t} \vec{v} .$$

Notice as t varies, $\vec{y}(t)$ is always a multiple of $\vec{v}$. In fact, since $e^{\lambda t}$ is always positive, $\vec{y}(t)$ is always a positive multiple of $C\vec{v}$. If $C>0$, $\vec{y}(t)$ is a positive multiple of $\vec{v}$, and if $C<0$, $\vec{y}(t)$ is a positive multiple of $-\vec{v}$.

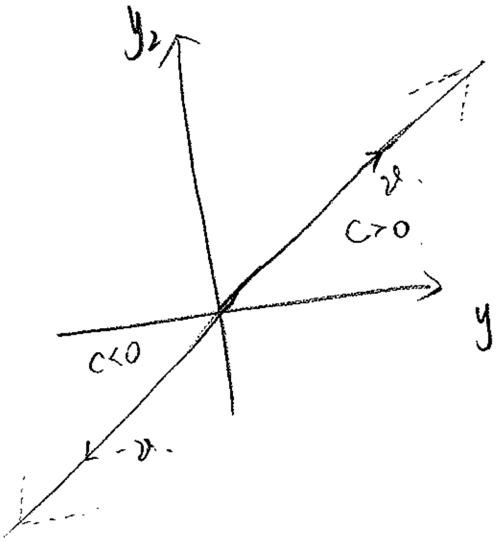
So if $\lambda > 0$, $e^{\lambda t}$ increases from 0 to ∞ as t increases from $-\infty$ to ∞.

While if $\lambda < 0$, $e^{\lambda t}$ decreases.

In either case, $\vec{y}(t)$ trace out the half-line consisting of positive multiples of $C\vec{v}$. Thus there are precisely two solution curves in the phase plane dependending on the sign of the constant C. Since these are half-lines, we will sometimes refer to exponential solutions as half-line solutions.

Stable vs Unstable.

When $\lambda > 0$, $e^{\lambda t}$ increases. On plase plane

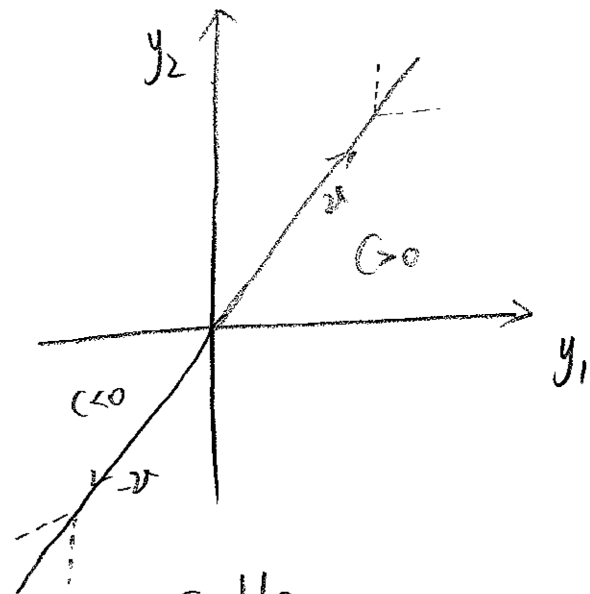


unstable.

the exponential solution tend away from the equilibrium pt at the origin as t increases and tend to origin as $t \to -\infty$.

Solution with this property are called $\boxed{\text{unstable solution}}$.

When $\lambda < 0$, $e^{\lambda t}$ decrease.



Stable

the exponential solution tend away from the equilibrium pt at the origin as t increases and tend to origin as $t \to -\infty$.

Solution with this property are called $\boxed{\text{stable solutions}}$

The dotted arrows indicate the direction of the flow as t increases.

Now the phase plane have many kinds of pictures, we give them names for referring to the equilibrium pt at the origin.

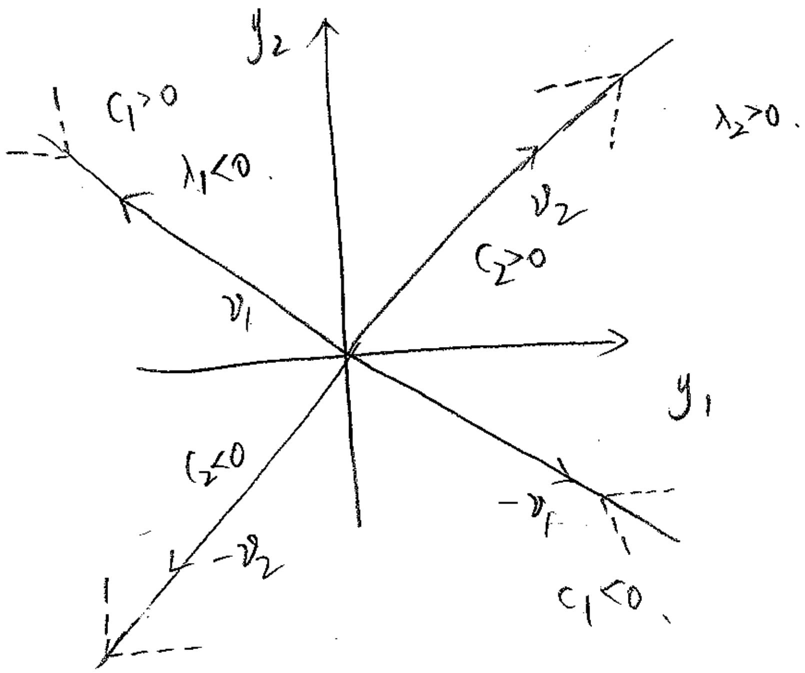
<u>Saddle pt.</u> (means the origin is saddle pt)

$(\lambda_1 < 0 < \lambda_2)$.

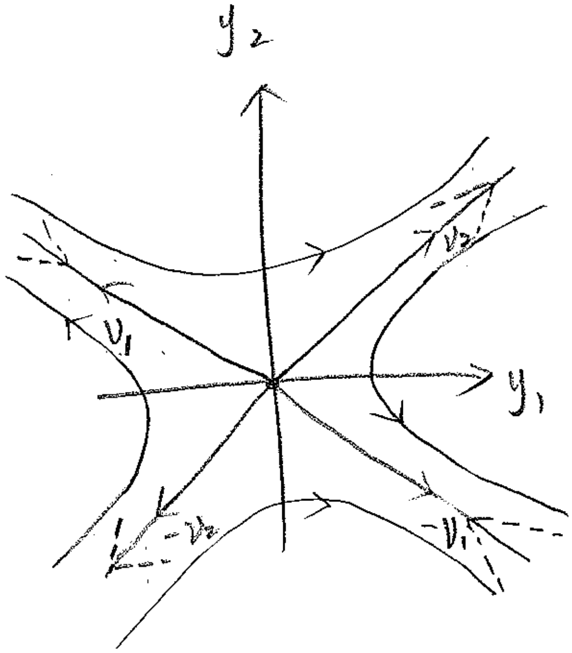
$\vec{y}(t) = C_1 e^{\lambda_1 t} \vec{v}_1 + C_2 e^{\lambda_2 t} \vec{v}_2$ are general solution.

Consider $\vec{y}(t) = C_1 e^{\lambda_1 t} \vec{v}_1$, $\vec{y}(t) = C_2 e^{\lambda_2 t} \vec{v}_2$.

exponential solutions



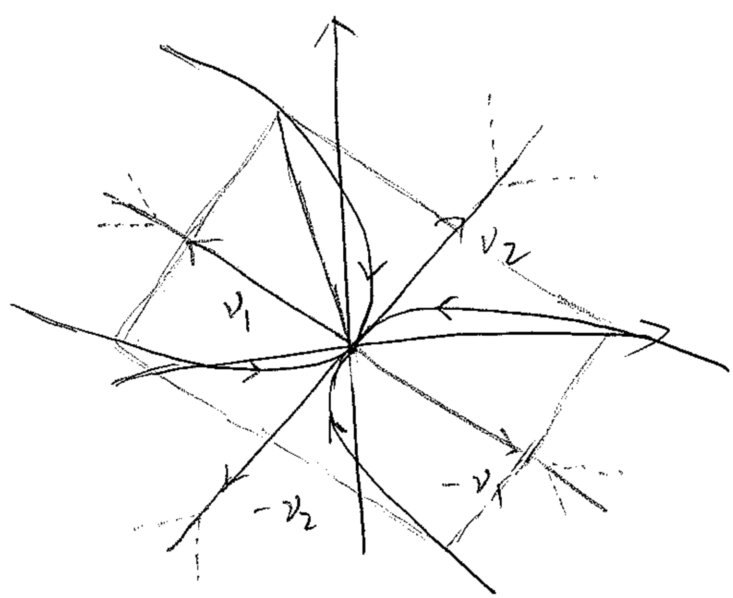
The origin acts like a saddle pt.



Notice if the solution curves were the altitude lines on a topolographic map, then the surface would have the shape of a saddle. This is the reason for the name saddle pt.

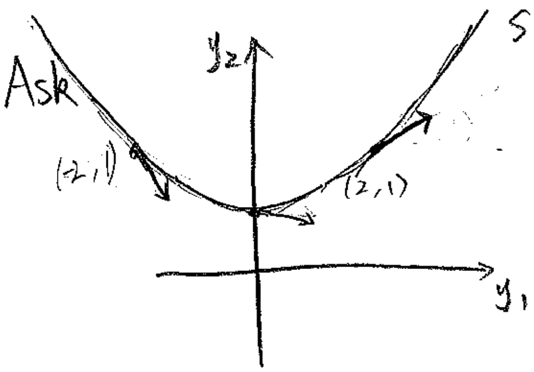
<u>Nodal sink.</u>

If $\lambda_1 < \lambda_2 < 0$.



In fact, we can draw direction field on plase plane y_1y_2-plane.

Ex. $\vec{y}' = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix} \vec{y}$. $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.



What is the slope of the direction at $(2,1)$?

Plug in. $(\vec{y})' = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ at pt $(2,1)$.

What is the slope of the direction at $(-2,1)$?

Plug in. $(\vec{y})' = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$ at pt $(-2,1)$.

What is the slope of the direction at $(0,\frac{1}{2})$?

Plug in $(\vec{y})' = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 \\ -\frac{1}{2} \end{pmatrix}$ at pt $(0,\frac{1}{2})$.

We can also solve it.

$$\vec{y}(t) = c_1 e^{-3t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Consider $\vec{y}(t) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$

$$= e^{\lambda_2 t} \left[c_1 e^{(\lambda_1 - \lambda_2) t} \vec{v}_1 + c_2 \vec{v}_2 \right]$$

As $t \to \infty$, $e^{(\lambda_1 - \lambda_2) t} \to 0$

$$\vec{y}(t) \longrightarrow e^{\lambda_2 t} \cdot c_2 \vec{v}_2 = c_2 e^{\lambda_2 t} \vec{v}_2 .$$

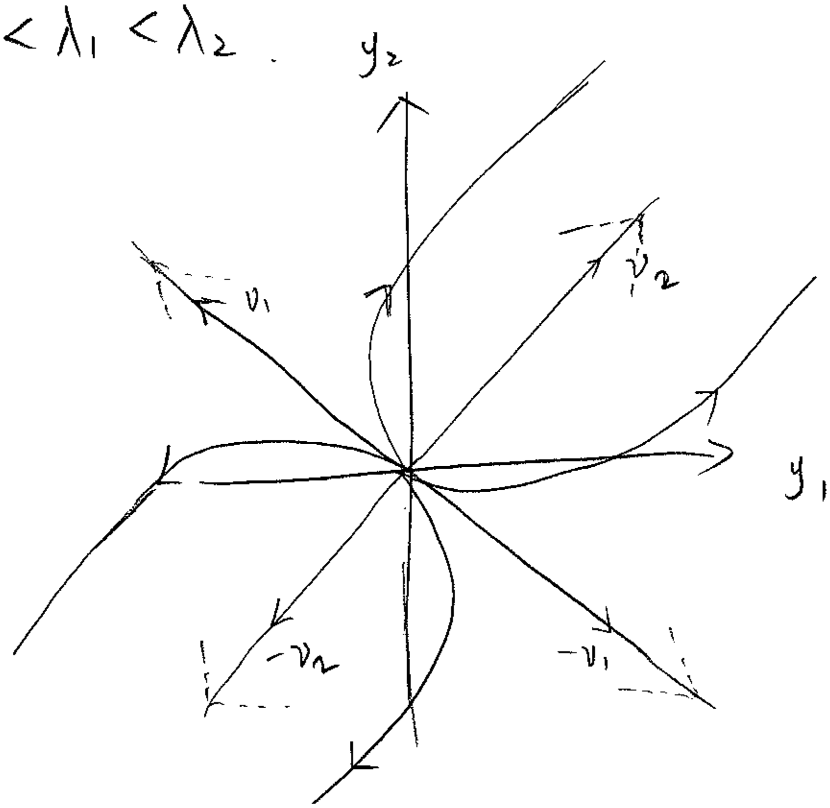
As $t \to -\infty$, $\vec{y}(t) = e^{\lambda_1 t} \left(c_1 \vec{v}_1 + c_2 e^{(\lambda_2 - \lambda_1) t} \vec{v}_2 \right)$

$$e^{(\lambda_2 - \lambda_1) t} \longrightarrow 0$$

$$\vec{y}(t) \longrightarrow e^{\lambda_1 t} c_1 \vec{v}_1 = c_1 e^{\lambda_1 t} \vec{v}_1 .$$

Nodal Source.

$0 < \lambda_1 < \lambda_2$.



$$\vec{y}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

$$= e^{\lambda_2 t}\left(c_1 e^{(\lambda_1-\lambda_2)t}\vec{v}_1 + c_2\vec{v}_2\right)$$

As $t \to \infty$, $\quad c_1 e^{(\lambda_1-\lambda_2)t}\vec{v}_1 \to 0.$

$$\vec{y}(t) \to c_2 e^{\lambda_2 t}\vec{v}_2 .$$

$$\vec{y}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

$$= e^{\lambda_1 t}\left(c_1\vec{v}_1 + c_2 e^{(\lambda_2-\lambda_1)t}\vec{v}_2\right)$$

As $t \to -\infty$, $\quad c_2 e^{(\lambda_2-\lambda_1)t}\vec{v}_2 \to 0$

$$\vec{y}(t) \longrightarrow c_1 e^{\lambda_1 t}\vec{v}_1$$

Complex Eigenvalues.

Recall For $\lambda = \alpha + i\beta$. and associated eigenvector $\vec{w} = \vec{v}_1 + i\vec{v}_2$.
our one complex solution is

$$\vec{y}(t) = e^{\lambda t}\vec{w}$$

$$= e^{(\alpha+i\beta)t}(\vec{v}_1 + i\vec{v}_2)$$

$$= e^{\alpha t}\left((\cos\beta t + i\sin\beta t)(\vec{v}_1 + i\vec{v}_2)\right)$$

$$= e^{\alpha t}(\cos\beta t\,\vec{v}_1 - \sin\beta t\,\vec{v}_2) + i e^{\alpha t}(\sin\beta t\,\vec{v}_1 + \cos\beta t\,\vec{v}_2).$$

So our fundamental set of real solutions are

$$\{e^{\alpha t}(\cos\beta t\vec{v}_1 - \sin\beta t\vec{v}_2),\ e^{\alpha t}(\sin\beta t\vec{v}_1 + \cos\beta t\vec{v}_2)\}.$$

so $\vec{y}(t) = c_1 e^{\alpha t}(\cos\beta t\vec{v}_1 - \sin\beta t\vec{v}_2) + c_2 e^{\alpha t}(\sin\beta t\vec{v}_1 + \cos\beta t\vec{v}_2)$

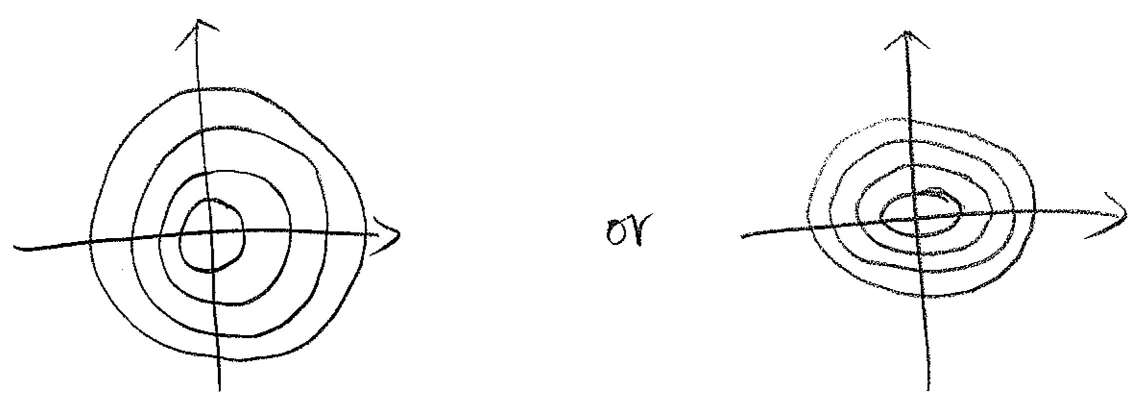
Center

If $\alpha = 0$. i.e the eigenvalues $\lambda = i\beta$.

then $\vec{y}(t) = c_1(\cos\beta t\vec{v}_1 - \sin\beta t\vec{v}_2) + c_2(\sin\beta t\vec{v}_1 + \cos\beta t\vec{v}_2)$.

The trigonometric fcns $\cos\beta t$, $\sin\beta t$ are both periodic with period $T = 2\pi/|\beta|$.

Consequently, the vector-valued fcn $\vec{y}(t)$ has the same property. This means that the solution trajectory is a closed curve, orbiting about the origin with period $T = 2\pi/|\beta|$.



The distinguishing characteristic of this type of equilibrium pt is that it is surrounded by closed solution curves.

An equilibrium point that has this property is called a center.

Ex. The system $\vec{y}' = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \vec{y}$.

Eigenvalue $\lambda = 2i$, $\quad \vec{w} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

complex Solution $\vec{z} = e^{\lambda t} \vec{w}$

$$= e^{2it} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2t + i \sin 2t \\ i \cos 2t - \sin 2t \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} + i \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}$$

so the fundamental set of real solutions are

$$\left\{ \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix}, \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} \right\}.$$

hence $\vec{y}(t) = c_1 \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} + c_2 \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}$.

so $y_1 = c_1 \cos 2t + c_2 \sin 2t$,

$y_2 = -c_1 \sin 2t + c_2 \cos 2t$.

$$\Rightarrow y_1^2 + y_2^2 = (c_1 \cos 2t + c_2 \sin 2t)^2 + (-c_1 \sin 2t + c_2 \cos 2t)^2$$

$$= c_1^2 \cos^2 2t + 2c_1 c_2 \cancel{\sin 2t \cos 2t} + c_2^2 \sin^2 2t$$

$$+ c_1 \sin^2 2t - 2c_1 c_2 \cancel{\sin 2t \cos 2t} + c_2^2 \cos^2 2t$$

$= C_1^2 + C_2^2.$

hence solution curve of (y_1, y_2) on phase plane y_1y_2-plane is a circle.

<u>Spiral Sink</u>

$\alpha < 0$. where eigenvalue $\lambda = \alpha + i\beta$, $\vec{w} = \vec{v}_1 + i\vec{v}_2$.

so $\vec{y}(t) = e^{\alpha t}\{C_1(\cos\beta t\, \vec{v}_1 - \sin\beta t\, \vec{v}_2) + C_2(\sin\beta t\, \vec{v}_1 + \cos\beta t\, \vec{v}_2)\}$.

Since if $t \to \infty$, $e^{\alpha t} \to 0$.

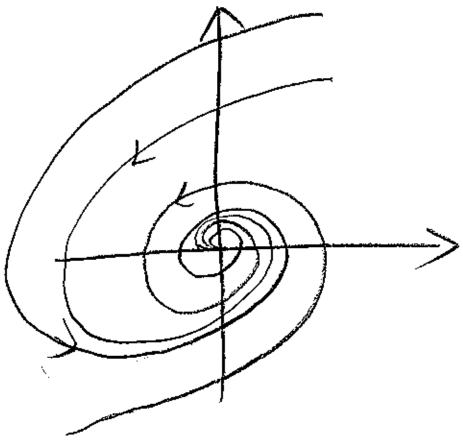
so $\vec{y}(t) \to 0$.

The $\{\ \}$ term is just what we see in center case.

so these terms are periodic with period $T = 2\pi/|\beta|$, and by themselves parametrize ellipses centered at the origin.

However, they are modified by the factor $e^{\alpha t}$.

Thus, while the solution curve circles the origin, it is being drawn toward the origin at the same time, resulting in a spiral motion. Since all solution curves spiral to the equilibrium pt at the origin, all solutions are stable.



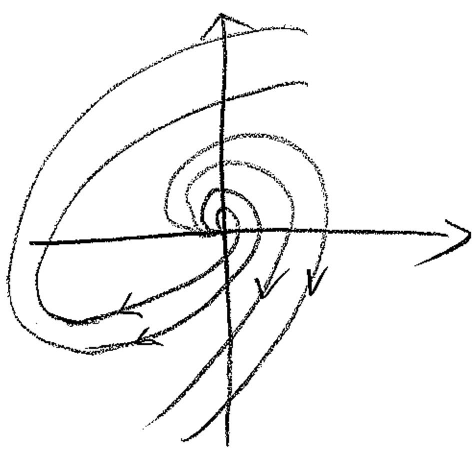
Spiral Source

$\alpha > 0$.

The solution $\vec{y}(t) = e^{\alpha t}\left(C_1(\cos\beta t\,\vec{v}_1 - \sin\beta t\,\vec{v}_2) + C_2(\sin\beta t\,\vec{v}_1 + \cos\beta t\,\vec{v}_2)\right)$.

this time $\alpha > 0$.

then as $t \to -\infty$, $\vec{y}(t)$ goes to zero.



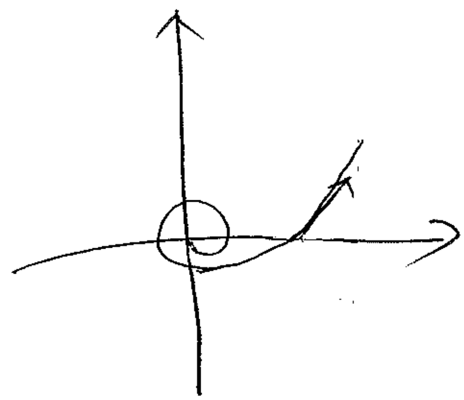
The direction of rotation

For spiral source.

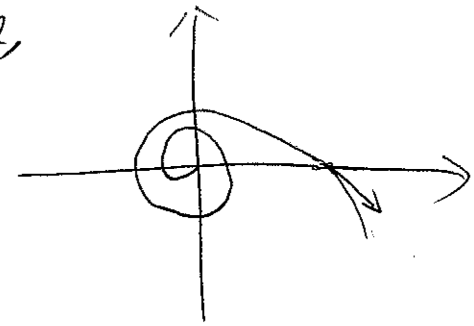
Generally, calculate the direction at pt (1,0).

$$\vec{y}'(t) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}.$$

If $a_{21} > 0$, this vector pts into upper half-plane, indicating the rotation is counterclockwise.



On the other hand, if $a_{21} < 0$, this vector pts into lower half-plane, indicating the rotation is clockwise.



Ex. $\vec{y}' = \begin{pmatrix} 1 & -4 \\ 2 & -3 \end{pmatrix} \vec{y}$. (For spiral sink)

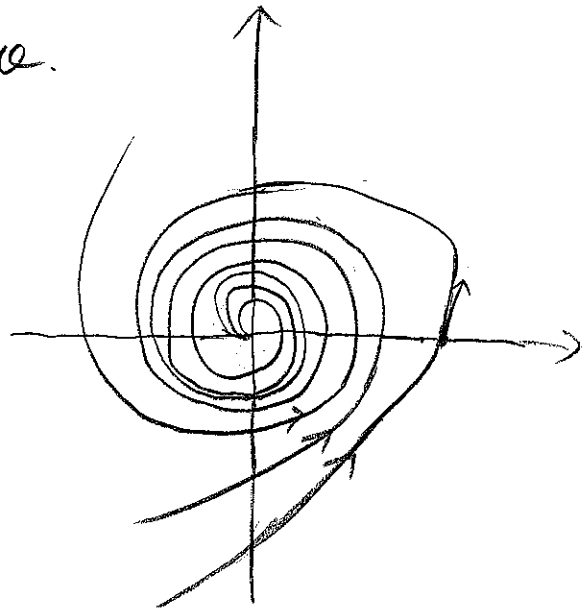
we have $\lambda = -1 + 2i$, $\vec{w} = \begin{pmatrix} 2 \\ 1-i \end{pmatrix}$.

since $\alpha < 0$. it's spiral sink.

how about rotation, at pt$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

direction is $\vec{y}' = \begin{pmatrix} 1 & 4 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ at pt $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

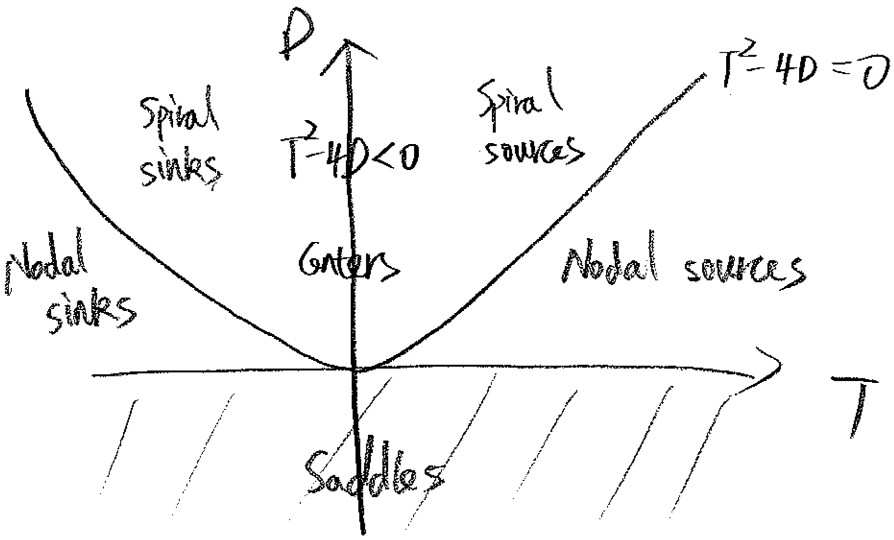
hence.



spiral . sink

The Trace - Derterminant plane.

Classification of equilibria.



$T^2-4D>0$.

$\lambda_1<\lambda_2$.

$$\lambda_1 = \frac{T-\sqrt{T^2-4D}}{2}, \quad \lambda_2 = \frac{T+\sqrt{T^2-4D}}{2}$$

$D=\lambda_1\lambda_2$.

$T=\lambda_1+\lambda_2$.

So for $T^2-4D>0$. $D<0$.

we have $\lambda_1<0<\lambda_2$. saddle case.

for $T^2-4D>0$, $D>0$.

① $T>0$, then $0<\lambda_1<\lambda_2$.

Nodal sources.

② $T<0$, then $\lambda_1<\lambda_2<0$

Nodal sinks.

for $T^2-4D<0$. complex case.

① $\alpha=0$. i.e $T=0$. center case.

② $\alpha<0$, i.e $T<0$, spiral sink.

③ $\alpha>0$, i.e $T>0$, spiral source.

Let's look specifically at the following five types:

- Saddle pt.
- Nodal sink.
- Nodal source.
- Spiral Sink.
- Spiral source.

Each of these types corresponds to a large open subset of TD-plane.

For that reason, call each of those five types $\boxed{\text{generic}}$.

Other $\boxed{\text{nongeneric}}$ types are

center + " $T^2-4D=0$ ".

Ex. Consider the system $y' = Ay$,

where $A = \begin{pmatrix} 4 & -3 \\ 15 & -8 \end{pmatrix}$.

Draw the solution curves on phase plane y_1y_2-plane.

Since $T = -4$, $D = -32 + 45 = 13$.

so $T < 0$, $D > 0$.

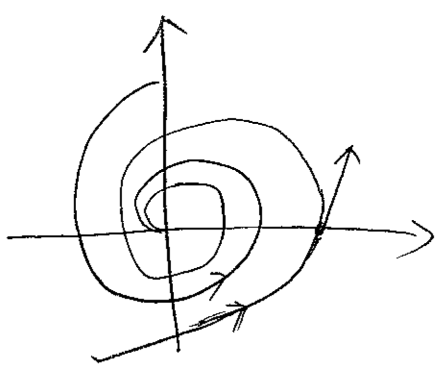
$T^2 - 4D = 16 - 52 < 0$.

so it's spiral sinks.

And we want to know the direction of rotation.

Consider slope at pt $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$\vec{y}' = \begin{pmatrix} 4 & -3 \\ 15 & -8 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 15 \end{pmatrix}.$$



Ex. Consider the system $y' = Ay$,

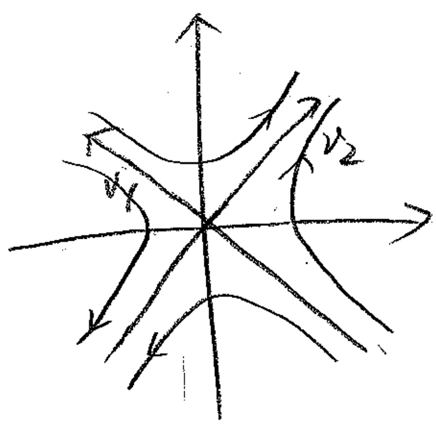
where $A = \begin{pmatrix} 8 & 5 \\ -10 & -7 \end{pmatrix}$.

Draw solution curves on y_1y_2-plane.

$T = 1, \quad D = 8\cdot(-7) + 50 = 2$

$T^2 - 4D = 1 - 8 = -7.$

so it's saddle pt.



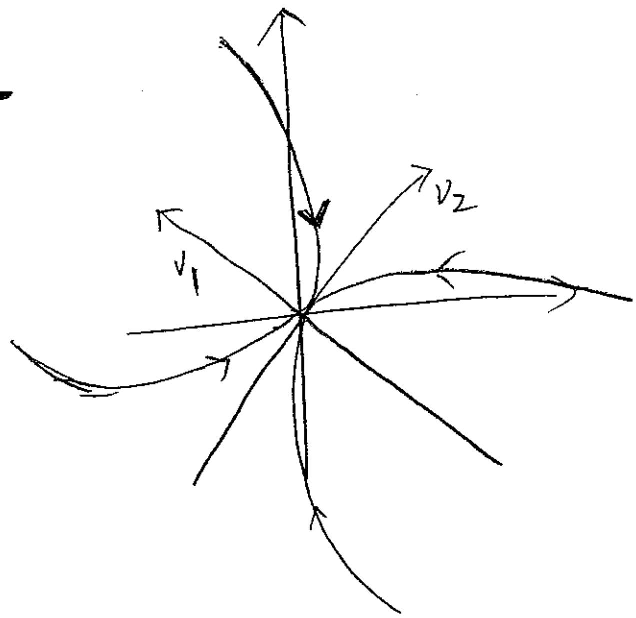
Ex. Consider the system $y' = Ay$,

where $A = \begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix}$.

$T = -3, \quad D = 2.$

$T^2 - 4D = 1 > 0.$

so it's Nodal sink.



Ex. Consider the system $y' = Ay$,

where $A = \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}$.

$T = 4$, $D = 3$

$T^2 - 4D = 4 > 0$

so it's Nodal source.