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Review

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1. $\frac{dy}{dt} = f(t)\,g(y)$ $\Leftarrow$ seperable equation.

or $y' = f(t)\,g(y)$.

Ex. $y' = e^x/(1+y)$, with $y(0)=1$.

$\Rightarrow \frac{dy}{dx} = e^x/(1+y)$

$\Rightarrow (1+y)\,dy = e^x\,dx$

$\Rightarrow \int (1+y)\,dy = \int e^x\,dx$

$\Rightarrow y + \frac{1}{2}y^2 = e^x + C$

$\Rightarrow y^2 + 2y - (2e^x + C) = 0$

$\Rightarrow y = \frac{-2 \pm \sqrt{4 + 4(2e^x + C)}}{2}$

$= -1 \pm \sqrt{2e^x + C}$.

Since $y(0)=1$.

so only $y = -1 + \sqrt{2e^x + C}$.

and $1 = -1 + \sqrt{2 + C}$

$\Rightarrow C = 2$.

so $y = -1 + \sqrt{2e^x + 2}$.

2. $x' = a(t)x + f(t)$. $\leftarrow$ linear equation.

$\Rightarrow x' - a(t)x = f(t)$. $\quad \mu = e^{-\int a(t)dt} \leftarrow$ integrating factor.

$\Rightarrow e^{-\int a(t)dt}(x' - a(t)x) = f(t) \cdot e^{-\int a(t)dt}$.

$\Rightarrow (e^{-\int a(t)dt}x)' = f(t) \cdot e^{-\int a(t)dt}$.

Ex. $x' = x\tan t + \sin t$, with $x(0) = 2$

so $\mu = e^{-\int \tan t\, dt} = e^{-\int \frac{\sin t\, dt}{\cos t}} = e^{\int \frac{d\cos t}{\cos t}} = e^{\ln(\cos t)} = \cos t$.

so $(x \cdot \cos t)' = \sin t \cdot \cos t$

$\Rightarrow x \cdot \cos t = \int \sin t \cos t\, dt$

$= \int \sin t\, d(\sin t)$

$= \frac{1}{2}\sin^2 t + c$

$\Rightarrow x = \frac{1}{2} \cdot \frac{\sin^2 t}{\cos t} + \frac{c}{\cos t}$.

Since $x(0) = 2$. $\Rightarrow 2 = \frac{1}{2} \cdot 0 + C$

$\Rightarrow C = 2$

hence $x = \frac{1}{2} \cdot \frac{\sin^2 t}{\cos t} + \frac{2}{\cos t}$.

3. Exact differential Equations.

Ex. Solve the equation $(xy-1)\,dx + (x^2-xy)\,dy = 0$.

$$P = xy-1, \quad Q = x^2-xy.$$

To check whether $\omega = P\,dx + Q\,dy$ is exact,

just need see whether $\partial P/\partial y = \partial Q/\partial x$

$$\partial P/\partial y = x,$$
$$\#$$
$$\partial Q/\partial x = 2x-y$$

We have next step by using integrating factor.

The form $P\,dx + Q\,dy$ has an integrating factor depending on one of the variables under the following conditions. • If $h = \frac{1}{Q}(\partial P/\partial y - \partial Q/\partial x)$ is a fcn of x only, then $\mu(x) = e^{\int h(x)\,dx}$ is an integrating factor. • If $g = \frac{1}{P}(\partial P/\partial y - \partial Q/\partial x)$ is a fcn of y only, then $\mu(y) = e^{-\int g(y)\,dy}$ is an integrating factor

$$h = \frac{1}{x^2-xy}\left(x-(2x-y)\right) = \frac{-x+y}{x^2-xy} = -\frac{1}{x}. \text{ is a fcn of } x \text{ only.}$$

so $\mu P\,dx + \mu Q\,dy$ is exact.

where $\mu = e^{\int h(x)dx} = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}.$

so $\mu P\,dx + \mu Q\,dy$

$$= \frac{1}{x}(xy-1)\,dx + (x-y)\,dy.$$

$$F(x,y) = \int \frac{1}{x}(xy-1)\,dx + \phi(y)$$

$$= \int (y - \frac{1}{x})\,dx + \phi(y)$$

$$= xy - \ln|x| + \phi(y)$$

$$\partial F/\partial y = x + \phi'(y) = \mu Q = x - y$$

$$\Rightarrow \phi'(y) = -y$$

$$\Rightarrow \phi(y) = -\frac{1}{2}y^2 + C.$$

hence $F(x,y) = xy - \ln|x| - \frac{1}{2}y^2 + C.$

so $y = y(x)$ is implicitly defined by

$$xy - \ln|x| - \frac{1}{2}y^2 + C = C'.$$

i.e. $xy - \ln|x| - \frac{1}{2}y^2 = C.$

4. Homogeneous Equation.

$$(x^2+y^2)\,dx - 2xy\,dy = 0.$$

Introduce $y = xv$.

$$\Rightarrow \quad x^2(1+v^2)\,dx - 2x^2v\,d(xv) = 0$$

$$\Rightarrow \quad (1+v^2)\,dx - 2v(x\,dv + v\,dx) = 0$$

$$\Rightarrow \quad (1-v^2)\,dx - 2vx\,dv = 0$$

$$\Rightarrow \quad \frac{dx}{2x} = \frac{2v}{1-v^2}\,dv$$

$$\Rightarrow \int \frac{dx}{2x} = \int \frac{2v\,dv}{1-v^2} = \int \frac{d(v^2)}{1-v^2}$$

$$\Rightarrow -\ln|1-v^2| = \frac{1}{2}\ln|x| + C$$

$$\Rightarrow \quad \ln|1-v^2| = -\frac{1}{2}\ln|x| + C$$

$$\Rightarrow \quad |1-v^2| = e^{-\frac{1}{2}\ln|x| + C}$$

$$= C \cdot \frac{1}{\sqrt{|x|}}.$$

$$\Rightarrow \quad 1-v^2 = C \cdot \frac{1}{\sqrt{|x|}}$$

$$\Rightarrow \quad v = \pm\sqrt{1 - C\cdot\frac{1}{\sqrt{|x|}}}.$$

$$\Rightarrow y = xv = \pm x\sqrt{1 - C\cdot\frac{1}{\sqrt{|x|}}}.$$

5. Solving system of equations.

$$-2x_1 - 3x_2 + x_3 + x_4 = 1$$

$$-4x_1 - 5x_2 + \quad + 4x_4 = 2$$

$$8x_1 + 9x_2 - 4x_3 - 9x_4 = 1$$

$$-8x_1 - 9x_2 + x_3 + 6x_4 = 0.$$

6. Find the basis of Span $\{v_1, v_2, v_3, v_4\}$.

where $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$, $v_4 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

Step 1: v_1 is included in basis.

Step 2: $v_2 \notin$ Span $\{v_1\}$.

so v_1, v_2 are in a basis.

Step 3: $v_3 \in$ Span $\{v_1, v_2\}$?

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 5 \\ 1 & 0 & 6 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 5 \\ 0 & -2 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 7 \end{pmatrix}.$$ No solution.

so $v_3 \notin$ Span $\{v_1, v_2\}$.

so v_1, v_2, v_3 are in a basis.

Since $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$.

Only 3 elts in a basis

$\Longrightarrow \{v_1, v_2, v_3\}$ is a basis.

7. Planar systems.

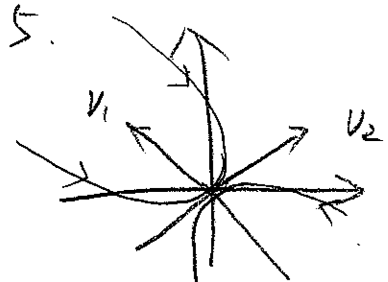
8. Only by Trace - Determinant, draw phase portrait.

(1) $A = \begin{pmatrix} -3 & -1 \\ -1 & -3 \end{pmatrix}$,

$T = -6$, $D = 9-4=5$.

$T^2 - 4D = 16 > 0$.

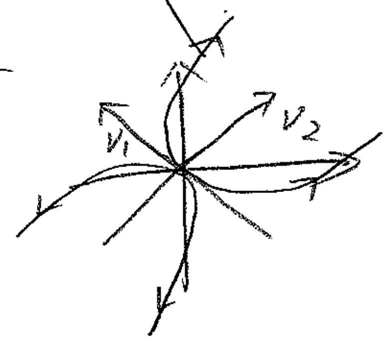
Nodal sink.



(2) $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$,

$T = 6$, $D = 9-4=5$

$T^2 - 4D = 16 > 0$



Nodal source.

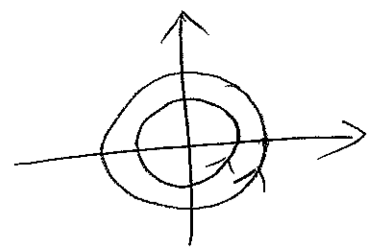
(3) $A = \begin{pmatrix} 4 & -10 \\ 2 & -4 \end{pmatrix}$.

$T = 0$, $D = -16 + 20 = 4$

$T^2 - 4D = -16 < 0$.

Center.

Direction of rotation?



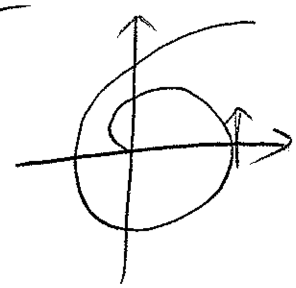
(4) $A = \begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix}$.

$T = -2$, $D = 0 + 2 = 2$

$T^2 - 4D = -4 < 0$.

Spiral sink.

Direction of rotation?



9. Higher-order equations with constant coefficients.

Ex. $y''' + 3y'' + 3y' + 1 = 0$.

The characteristic polynomial is

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1$$
$$= (\lambda+1)^3.$$

so -1 is the only eigenvalue with multiplicity 3.

the fundamental set of solutions will be

$$\{e^{-t}, te^{-t}, t^2e^{-t}\}.$$

10. Inhomogeneous equations: the method of undetermined coefficients.

$$y'' + 3y' + 2y = 3e^{-4t} + 1. \quad y(0) = 1, \ y'(0) = 0.$$

<u>Remember</u>: $3e^{-4t}+1$ should be 2 parts.

Part 1: $y'' + 3y' + 2y = 3e^{-4t}$.

Assume $y = ae^{-4t}$.

$\Rightarrow y' = -4ae^{-4t}$, $y'' = 16ae^{-4t}$.

so $y'' + 3y' + 2y = 16ae^{-4t} - 12ae^{-4t} + 2ae^{-4t}$

$= 6ae^{-4t} = 3e^{-4t}$.

$\Rightarrow a = \frac{1}{2}$.

So $y = \frac{1}{2}e^{-4t}$.

Part 2: $y'' + 3y' + 2y = 1$.

so assume $y = c$.

$\Rightarrow \quad 2c = 1 \quad \Rightarrow c = \frac{1}{2}$.

so $y_p = \frac{1}{2}e^{-4t} + \frac{1}{2}$.

General solution of $y''+3y'+2y=0$ $\quad \lambda^2+3\lambda+2=0 \Rightarrow \lambda=-1, \lambda=-2.$

is $y=c_1e^{-t}+c_2e^{-2t}$

so general solution of $y''+3y'+2y=3e^{-4t}+1$

are $y=c_1e^{-t}+c_2e^{-2t}+\frac{1}{2}e^{-4t}+\frac{1}{2}.$

since $y(0)=1$

$\Rightarrow \quad c_1+c_2+\frac{1}{2}+\frac{1}{2}=1 \quad \Rightarrow c_1+c_2=0$

Since $y'(0)=0.$

$y'(t)=-c_1e^{-t}-2c_2e^{-2t}-2e^{-4t}$

$\Rightarrow \quad -c_1-2c_2-2=0$

so $\quad c_1=-2, \ c_1=2.$

so $y=-2e^{-t}+2e^{-2t}+\frac{1}{2}e^{-4t}+\frac{1}{2}.$

11. Variation of Parameters.

Ex. $y''+3y'+2y = te^t$.

The forcing term is not in the cases we can solve.

we can use variation of Parameter method.

Step 1: $y''+3y'+2y=0$.

$\Rightarrow \lambda^2+3\lambda+2=0 \Rightarrow \lambda=-2, \lambda=-1$

so $y_1 = e^{-2t}$, $y_2 = e^{-t}$.

Step 2: $y_p = v_1 \cdot y_1 + v_2 y_2$

$= v_1 e^{-2t} + v_2 e^{-t}$.

$y_p' = (-2v_1 e^{-2t} - v_2 e^{-t}) + (v_1' e^{-2t} + v_2' e^{-t})$

Assume $v_1' e^{-2t} + v_2' e^{-t} = 0$.

$y_p'' = (4e^{-2t} v_1 + v_2 e^{-t}) + (-2v_1' e^{-2t} - v_2' e^{-t})$

Now $y_p'' + 3y_p' + 2y_p$

$= (-2v_1' e^{-2t} - v_2' e^{-t}) = te^t$.

so $\begin{cases} e^{-2t} v_1' + e^{-t} v_2' = 0 \\ -2e^{-2t} v_1' - e^{-t} v_2' = te^t. \end{cases}$

so $\begin{pmatrix} e^{-2t} & e^{-t} & 0 \\ -2e^{-2t} & -e^{-t} & te^{t} \end{pmatrix}$

$\xRightarrow{②+2①} \begin{pmatrix} e^{-2t} & e^{-t} & 0 \\ 0 & e^{-t} & te^{t} \end{pmatrix}$

$\Rightarrow \quad v_2' = te^{2t}, \quad v_1' = -te^{3t}.$

so $\quad v_1 = -\int te^{3t}dt = -\frac{1}{3}e^{3t} \cdot t + \int \frac{1}{3}e^{3t}dt$

$= -\frac{1}{3}te^{3t} + \frac{1}{9}e^{3t}.$

$v_2 = \int te^{2t}dt = \frac{1}{2}e^{2t} \cdot t - \int \frac{1}{2}e^{2t}dt$

$= \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t}.$

so $y_p = v_1 y_1 + v_2 y_2$

$= \left(-\frac{1}{3}te^{3t} + \frac{1}{9}e^{3t}\right)e^{-2t} + \left(\frac{1}{2}te^{2t} - \frac{1}{4}e^{2t}\right)e^{-t}$

$= -\frac{1}{3}te^{t} + \frac{1}{9}e^{t} + \frac{1}{2}te^{t} - \frac{1}{4}e^{t}$

$= \frac{1}{6}te^{t} - \frac{5}{36}e^{t}.$

so general solution is $y = \frac{1}{6}te^{t} - \frac{5}{36}e^{t} + C_1 e^{-2t} + C_2 e^{-t}.$

12. Exponential of a Matrix.

Type 1: If I know for $A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$.

$$\exists \quad P = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \quad \text{s.t} \quad PAP^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

how can we get e^A?

$$\text{Since} \quad e^{PAP^{-1}} = Pe^{A}P^{-1}$$

$$\| $$

$$e^{\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}}$$

$$\|$$

$$\begin{pmatrix} e^2 & \\ & e^1 \end{pmatrix}$$

so $e^A = P^{-1}\begin{pmatrix} e^2 & \\ & e^1 \end{pmatrix} P$.

Step 1: Find P^{-1}.

$$\left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{②+①} \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{①+②} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

so $P^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Step 2:

$$e^A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^2 & \\ & e^1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2e^2 & e^1 \\ e^2 & e^1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2e^2 - e^1 & -2e^2 + 2e^1 \\ e^2 - e^1 & -e^2 + 2e^1 \end{pmatrix}.$$

Type 2: Determine the smallest k s.t $(A - \lambda I)^k = 0$. then calculate e^{tA}.

$$A = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & -1 \\ -2 & 4 & -3 \end{pmatrix}.$$

Step 1: Find the λ.

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda+1 & 0 & 0 \\ 1 & \lambda-1 & 1 \\ 2 & -4 & \lambda+3 \end{pmatrix}$$

$$= (\lambda+1)\,[(\lambda-1)(\lambda+3) + 4]$$

$$= (\lambda+1)(\lambda^2 + 2\lambda + 1) = (\lambda+1)^3.$$

so $\lambda = -1$.

so $A+I=\begin{pmatrix}0&0&0\\-1&2&-1\\-2&4&-2\end{pmatrix}\neq 0$ $k\neq 1$.

$$(A+I)^2=\begin{pmatrix}0&0&0\\-1&2&-1\\-2&4&-2\end{pmatrix}\begin{pmatrix}0&0&0\\-1&2&-1\\-2&4&-2\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}.$$

so $k=2$.

hence $$e^{tA}=e^{-t}\left(I+t(A+I)\right)$$

$$=e^{-t}\cdot\left(\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}+t\begin{pmatrix}0&0&0\\-1&2&-1\\-2&4&-2\end{pmatrix}\right)$$

$$=e^{-t}\begin{pmatrix}1&0&0\\-t&2t+1&-t\\-2t&4t&-2t+1\end{pmatrix}.$$

13. Higher-dimensional system.

$$\vec{y}' = A\vec{y}, \quad \text{where } A = \begin{pmatrix} -1 & -2 & 1 \\ 0 & -4 & 3 \\ 0 & -6 & 5 \end{pmatrix}.$$

The characteristic equation is

$$\lambda^3 - 3\lambda - 2 = (\lambda+1)^2(\lambda-2) = 0.$$

so the eigenvalues are -1 and 2, with -1 having multiplicity 2.

The eigenspace for the eigenvalue $\lambda_1 = 2$ is

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Hence $\vec{y}_1(t) = e^{tA}\vec{v}_1 = e^{2t}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$

For the eigenvalue $\lambda_2 = -1$, we have

$$A + I = \begin{pmatrix} 0 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & -6 & 6 \end{pmatrix},$$

$\dim \operatorname{Null}(A+I) = 1$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. $\quad y_2(t) = e^{tA}\vec{v}_2 = e^{-t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$

We need our third vector $\vec{v}_3$.

Find v_3 s.t

$$(A+I)v_3 = v_2.$$

so $\begin{pmatrix} 0 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & -6 & 6 \end{pmatrix} v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 0 & -2 & 1 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & -6 & 6 & 0 \end{pmatrix}$$

$$\xrightarrow{③-2②} \begin{pmatrix} 0 & -2 & 1 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\xrightarrow{②-\frac{3}{2}①} \begin{pmatrix} 0 & -2 & 1 & 1 \\ 0 & 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \quad x_3 = -1, \ x_2 = 1, \ x_1 = t.$

so choose $v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$

so $y_3(t) = \boxed{e^{tA}\vec{v}_3 = e^{-t}\left(v_3 + t(A+I)v_3\right)}$

$$= e^{-t}(v_3 + t v_2).$$

$$= e^{-t}\begin{pmatrix} -t \\ 1 \\ 1 \end{pmatrix}.$$

14. Linearly independence of fcns or vectors of fcns.

Type 1: $f_1(t) = e^t + \cos t$, $f_2 = e^t$, $f_3 = e^{2t}$.

Whether they are linearly independent.

$$W(t) = \det \begin{pmatrix} e^t + \cos t & e^t & e^{2t} \\ e^t - \sin t & e^t & 2e^{2t} \\ e^t - \cos t & e^t & 4e^{2t} \end{pmatrix}$$

For $t=0$, $W(0) = \det \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 4 \end{pmatrix}$

$$\xrightarrow{②-\frac{1}{2}①} = \det \begin{pmatrix} 2 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 4 \end{pmatrix}$$

$$\xrightarrow{③-2②} = \det \begin{pmatrix} 2 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 \end{pmatrix} \neq 0.$$

so Yes!

Type 2:

Whether $\vec{f_1} = e^{2t}\begin{pmatrix}1\\0\\1\end{pmatrix}$, $\vec{f_2} = e^{3t}\begin{pmatrix}0\\1\\1\end{pmatrix}$, $\vec{f_3} = e^{4t}\begin{pmatrix}2\\1\\2\end{pmatrix}$

linearly independent?

$$W(t) = \det(\vec{f_1}, \vec{f_2}, \vec{f_3})$$

$$= \det\begin{pmatrix} e^{2t} & 0 & 2e^{4t} \\ 0 & e^{3t} & e^{4t} \\ 1 & e^{3t} & 2e^{4t} \end{pmatrix}$$

$$W(0) = \det\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} = 0.$$

so No!