

Research Statement

Ryan Dunning, rdunning@rice.edu

1 Introduction

The main subject of my research is the Möbius energy of curves in Euclidean space, which is a variational problem with deep topological roots and potential applications to biological settings. The notion of the energy of a curve was introduced by [S] and [F], and a family of so-called “knot energies” was developed extensively by [O1] and [O2] as an attempt to determine an “optimally embedded” representative of a given knot type.

Further work in [FHW] showed that one of O’Hara’s knot energies, the “Möbius energy,” does yield such a result for prime knots, namely that there exists a simple closed curve in \mathbb{R}^3 whose Möbius energy is less than or equal to the Möbius energy of any other closed curve of the same knot type. The fact that the Möbius energy is self-repulsive (it blows up for a sequence of curves approaching a curve with a self-intersection) is a key component of the proof of this minimization result.

My work ([D]) uses this energy to study the behavior of optimally immersed self-intersecting curves in the Euclidean plane. Because any such curve has infinite Möbius energy, the exact methods of [FHW] do not apply. Instead, an approach in the style of a Ginzburg-Landau problem (see [BBH]) better serves the problem. I make appropriate parameter-dependent renormalizations of the Möbius energy so as to remove the singularity, solve the minimization problem associated to that parameter, then let the parameter degenerate. This framework allows for the careful selection of an optimally immersed curve as the limit (away from self-intersections) of minimizers of the renormalized energy.

2 Möbius Energy and its Renormalizations

Definition 2.1. [FHW] Let γ be a curve in \mathbb{R}^3 given by $\gamma(x)$, where x -values are taken from some subinterval X of \mathbb{R} or of \mathbb{S}^1 . Also assume that $|\dot{\gamma}|$ is locally integrable, where $|\cdot|$ is the standard Euclidean norm. Denote by $D(\gamma(x), \gamma(y))$ the length along the curve γ of the shorter arc between $\gamma(x)$ and $\gamma(y)$. The (Möbius) energy $E(\gamma)$ is

$$E(\gamma) = \iint_{X \times X} \left(\frac{1}{|\gamma(x) - \gamma(y)|^2} - \frac{1}{D(\gamma(x), \gamma(y))^2} \right) |\dot{\gamma}(x)| |\dot{\gamma}(y)| dx dy.$$

The first term is called the *principal term*, and it models Coulomb’s repulsive force between positive point charges at $\gamma(x)$ and $\gamma(y)$. The second term is called the *counter term*, as it cancels out the infinity caused by the principal term along the diagonal of $X \times X$.

It is clear from this definition that the energy is non-negative and invariant under rigid motion, rescaling, and reparametrization. A strong result of [FHW] is that it is also invariant under Möbius transformations of $\mathbb{R}^3 \cup \{\infty\}$, thereby giving Möbius energy its name.

As mentioned before, finding a curve with minimal Möbius energy among a class of self-intersecting curves is meaningless as stated, since these curves have infinite energy. Thus the optimal immersion problem requires some means of calculating the energy away from self-intersections.

To begin this process, let $\gamma : X \rightarrow \mathbb{R}^2$ be a self-intersecting closed curve. Assume that γ has unit length and intersects itself only once, so that it belongs to one of only two possible planar isotopy classes, I_1 or I_2 . Let I_1 be the class containing figure eight shapes, and let I_2 be the other class (see Figure 1).

Denote by $u, v \in X$ the distinct times that yield $\gamma(u) = \gamma(v)$. Let $\lambda(\gamma) = D(\gamma(u), \gamma(v))$, the portion of the length of γ taken up by the shorter of its two distinct “loops.” Note that $0 < \lambda \leq 1/2$.

Definition 2.2. With γ, u, v , and λ as above, choose $\delta_0 \ll \lambda$. Let S_{δ_0} be the set of all pairs $(x, y) \in X \times X$ such that either

1. $D(\gamma(x), \gamma(u)) \leq \delta_0$ and $D(\gamma(y), \gamma(v)) \leq \delta_0$ or
2. $D(\gamma(x), \gamma(v)) \leq \delta_0$ and $D(\gamma(y), \gamma(u)) \leq \delta_0$.

Thus S_{δ_0} is the set of (x, y) such that x and y are on separate arcs of the curve γ but both within arclength δ_0 of the self-intersection. Define the δ_0 -renormalized (Möbius) energy $E_{\delta_0}(\gamma)$ to be the energy of γ taken outside of this troublesome set, that is

$$E_{\delta_0}(\gamma) = \iint_{(X \times X) \setminus S_{\delta_0}} \left(\frac{1}{|\gamma(x) - \gamma(y)|^2} - \frac{1}{D(\gamma(x), \gamma(y))^2} \right) |\dot{\gamma}(x)| |\dot{\gamma}(y)| dx dy.$$

Note that E_{δ_0} is invariant under rigid motion and reparametrization.

Definition 2.3. Choose $\lambda \in (0, \frac{1}{2}]$. Let $I_j \in \{I_1, I_2\}$ be an isotopy class of closed curves in \mathbb{R}^2 with one self-intersection. Define the class $\mathcal{F}_{j, \lambda, \delta_0} \subset I_j$ to be the curves $\gamma \in I_j$ satisfying the following.

1. The curve γ has unit length.
2. The curve γ intersects itself at the origin.
3. Within arclength δ_0 of the self-intersection, γ is linear.
4. The collection of these four linear subarcs of γ enjoy two-fold rotational symmetry about the origin.

5. The short loop length $\lambda(\gamma)$ equals the fixed λ .

Condition 2 in the above definition is not restrictive, due to the translation invariance of the energy. Conditions 3 and 4 together imply that the curve is a skewed plus sign at the self-intersection.

Theorem 2.4. *For every $0 < \delta_0 \ll \lambda$, there exists a curve $\gamma_{\delta_0} \in \mathcal{F}_{j,\lambda,\delta_0}$ such that $E_{\delta_0}(\gamma_{\delta_0}) \leq E_{\delta_0}(\gamma)$ for all $\gamma \in \mathcal{F}_{j,\lambda,\delta_0}$.*

The proof of this theorem combines a standard compactness argument with the following variation of a result of [O2].

Lemma 2.5. *Let $\gamma : X \rightarrow \mathbb{R}^2$ be a curve in the class I_1 or I_2 with unit speed and unit length. Choose $\delta_0 \ll \lambda = \lambda(\gamma)$. If the renormalized energy $E_{\delta_0}(\gamma)$ is finite, then there exists a constant*

$$L \leq \frac{24}{\lambda^2} \exp \left[\frac{49}{40} E_{\delta_0}(\gamma) \right]$$

which satisfies

$$\frac{|x - y|}{L} \leq |\gamma(x) - \gamma(y)| \leq |x - y| \tag{1}$$

for all $(x, y) \in (X \times X) \setminus S_{\delta_0}$.

Because there is no uniform bound on $E_{\delta_0}(\gamma_{\delta_0})$ as $\delta_0 \rightarrow 0$, Theorem 2.4 does not give an immediate candidate for an optimally immersed curve γ in some class of curves. An estimate on the energy of E_{δ_0} -minimizing curves γ_{δ_0} as $\delta_0 \rightarrow 0$, calculated some arbitrarily small fixed distance away from the self-intersection, is necessary to show that the standard Möbius energy concentrates near the self-intersection. To this end, choose any positive $\eta \ll \lambda$, and let $\{\delta_i\}_{i=1}^{\infty}$ be a sequence such that $\delta_i \rightarrow 0$ as $i \rightarrow \infty$ and $0 < \delta_i < \eta$ for all i . By analog with Definition 2.2, let $E_{\eta,\delta}$ be the energy integral over pairs of points $(x, y) \in \mathcal{S}_\eta \setminus \mathcal{S}_\delta$. Note that

$$E_\eta(\gamma_{\delta_i}) = E_{\delta_i}(\gamma_{\delta_i}) - E_{\eta,\delta_i}(\gamma_{\delta_i}). \tag{2}$$

The problem then is to use this inequality to construct a bound on $E_\eta(\gamma_{\delta_i})$ dependent on the fixed λ and η , but independent of i . Because γ_{δ_i} was chosen to minimize E_{δ_i} , one only needs to calculate an upper bound on $E_{\delta_i}(\gamma_c)$ for some candidate $\gamma_c \in \mathcal{F}_{j,\lambda,\delta_i} \subset I_j$. Note that this quantity *will* depend on δ_i . At present, assume that $I_j = I_1$, the class of curves isotopic to the figure eight shape, and let $\gamma_c \in \mathcal{F}_{1,\lambda,\delta_i}$ be the candidate curve in the figure, noting that r_1 and r_2 depend only on λ .

Lemma 2.6. *The δ_i -renormalized energy of the candidate curve γ_c satisfies*

$$E_{\delta_i}(\gamma_c) \leq C(\lambda) - C(\lambda, \delta_i) + \pi \log \frac{r_1 r_2}{\delta_i^2}, \tag{3}$$

where $C(\lambda) < \infty$ is a constant dependent only upon λ , and $C(\lambda, \delta_i) < \infty$ is a constant dependent only upon λ and δ_i .

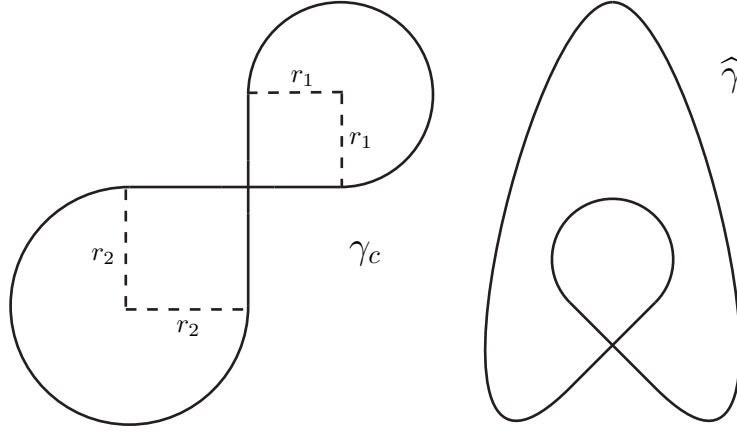


Figure 1: The $\frac{3}{4}$ -circles curve $\gamma_c \in \mathcal{F}_{1,\lambda,\delta_0} \subset I_1$ and a curve $\hat{\gamma} \in I_2$.

Because of the structure of the curve class $\mathcal{F}_{j,\lambda,\delta_i}$, straightforward calculus reveals that E_{η,δ_i} achieves its minimal value over $\mathcal{F}_{j,\lambda,\delta_i}$ for a curve $\gamma_{\eta,\delta_i} \in \mathcal{F}_{j,\lambda,\delta_i}$ whose self-intersection is a sufficiently large straight-line orthogonal crossing. Thus (2) becomes

$$E_{\eta}(\gamma_{\delta_i}) \leq E_{\delta_i}(\gamma_c) - E_{\eta,\delta_i}(\gamma_{\eta,\delta_i}). \quad (4)$$

Because the curve $\gamma_c \in \mathcal{F}_{1,\lambda,\delta_i}$ was constructed with such a crossing, a computation almost identical to (3) shows that the terms on the right hand side of (4) have the same asymptotic dependence on δ_i . This makes the quantity $E_{\eta}(\gamma_{\delta_i})$ uniformly bounded above by a constant dependent only on η and λ over all i . A standard diagonalization argument then asserts the existence of an optimally immersed curve $\gamma^1 \in \mathcal{F}_{1,\lambda,0}$.

3 Future Work

In the near future, I aim to modify the gradient flow methods of [H] in order to investigate the regularity of optimally immersed curves. It is reasonable to expect that an optimal immersion is of class C^∞ , as [H] shows that locally optimal embedded curves are of class C^∞ . Similar work may also be helpful in proving the conjecture that optimally immersed curves intersect themselves with orthogonal tangents.

There are many reasonable generalizations to make to my work. If the value λ is allowed to vary within the problem, then it is likely that the optimal immersion $\gamma^1 \in \mathcal{F}_1$ satisfies $\lambda(\gamma^1) = 1/2$, while the optimal immersion $\gamma^2 \in \mathcal{F}_2$ will satisfy $\lambda(\gamma^2) < 1/2$. Theorem 2.4 and Lemma 2.5 have immediate analogs for classes of unit length curves with multiple self-intersections, but more care is needed to salvage these results if the restriction on length is lifted as well.

Furthermore, the isotopy classes can be relaxed from planar curves to spatial curves.

One or more of these generalizations, carefully selected, have the potential for interesting and accessible research projects for small undergraduate groups. In particular, the applications of knot energies in biology (see, for example, [BDKMS]) also lend the subject to interdisciplinary projects.

Another well-known model of the energy of a knot is the ropelength (see [BO]). I am interested in translating my problem into this setting and comparing the techniques needed, as well as the results obtained.

References

- [BDKMS] Jan Bednar, Jacques Dubochet, Vsevolod Katritch, Didier Michoud, Andrzej Stasiak, *Electrophoretic mobility of DNA knots*, *Nature*, **384** (1996), 122.
- [BBH] Fabrice Bethuel, Haïm Brezis, Frédéric Hélein, *Ginzburg-Landau Vortices*, *Progress in Nonlinear Differential Equations and Their Applications*, **13**, Birkhäuser, Boston, (1994).
- [BO] Gregory Buck, Jeremy Orloff, *A simple energy function for knots*, *Topology Appl.*, **61** (1995), 205-214.
- [D] Ryan Dunning, *Asymptotics under self-intersection for minimizers of self-avoiding energies*, Ph.D. Thesis, Rice University, in preparation.
- [FHW] Michael Freedman, Zheng-Xu He, Zhenghan Wang, *Möbius energy of knots and unknots*, *Ann. of Math. (2)*, **139** (1994), 1–50.
- [F] S. Fukuhara, *Energy of a knot*, *A Fête of Topology* (Y. Matsumoto, T. Mizutani, S. Morita, eds.), Academic Press, Boston, (1988), 443–451.
- [H] Zheng-Xu He, *The Euler-Lagrange equation and heat flow for the Möbius energy*, *Comm. Pure Appl. Math.*, **53** (2000), 399–431.
- [O1] Jun O’Hara, *Energy of a knot*, *Topology*, **30** (1991), 241–247.
- [O2] Jun O’Hara, *Family of energy functionals of knots*, *Topology Appl.*, **48** (1992), 147–161.
- [S] M. Sakuma, *Problem no. 8 (in Japanese)*, *The collection of problems on “Low dimensional topology and related matters”* (S. Kojima, S. Negami, eds.), (1987), 7.