So at the end of class today, we wanted to compute some horizontal asymptotes. I told you that there are two main tricks that will get the job done - but I didn’t do the best job of showing you either! So... here are a few examples to fill in the gaps.

**Example 1:** Find \( \lim_{x \to \infty} f(x) \) where \( f(x) = \frac{3x^5 + 4}{2x^6 - 7x + 3} \)

This is a rational function, and we see that if \( g(x) = 3x^5 + 4 \) and \( h(x) = 2x^6 - 7x + 3 \), then the long term behavior of the function is dictated by the highest power. The highest power of \( x \) runs the show. This is because as \( x \) gets very large, the difference between \( x^n \) and \( x^{n-1} \) for \( n > 0 \) is very big (to prove this to yourself, try \( n = 2 \) and \( x = 100 \)). So we can deduce that \( \lim_{x \to \infty} g(x) \) only depends on \( \lim_{x \to \infty} x^5 \). Since given any (very big) value \( L \), we can find some \( x \) such that \( x^5 \) is close to \( L \), \( \lim_{x \to \infty} x^5 = \infty \).

The argument is exactly the same for \( h(x) \), giving \( \lim_{x \to \infty} h(x) = \infty \).

Side note: that if we were concerned with \( \lim_{x \to -\infty} f(x) \), we’d have to look at \( \lim_{x \to -\infty} g(x) \), which depends on \( \lim_{x \to -\infty} x^5 = -\infty \). Hence \( \lim_{x \to -\infty} g(x) = -\infty \). On the other hand Since \((-1)^{\text{even power}} = 1\) raised to an even power is positive, we get \( \lim_{x \to -\infty} h(x) = \infty \).

We know that the limits as \( x \to \infty \) of both \( g(x) \) and \( h(x) \) exist by the above argument, so we can use the limit laws we learned in section 2.3.

Now, as I said above, the biggest power determines the behavior of the function. So we want to make the little, inconsequential powers “go away.” To do this, we use the first trick I mentioned in class: divide the numerator and denominator by the biggest power of \( x \) found in the denominator.

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\frac{1}{x^5}(3x^5 + 4)}{\frac{1}{x^6}(2x^6 - 7x + 3)}
\]

Simplifying, we obtain

\[
= \lim_{x \to \infty} \frac{3x^5 + 4}{2x^6 - 7x + 3}
\]

\[
= \lim_{x \to \infty} \frac{3x/ x^5 + 4/ x^6}{2x^6 - 7x/ x^5 + 3/ x^5}
\]

\[
= \lim_{x \to \infty} \frac{3/x + 4/x^6}{2 - 7/x^5}
\]
Now we can use the quotient law
\[ \lim_{x \to \infty} f(x) = \frac{\lim_{x \to \infty} 3/x + 4/x^6}{\lim_{x \to \infty} 2 - 7/x^5} \]

Then the sum law tells us
\[ \lim_{x \to \infty} f(x) = \frac{\lim_{x \to \infty} 3/x + \lim_{x \to \infty} 4/x^6}{\lim_{x \to \infty} 2 - \lim_{x \to \infty} 7/x^5} \]

By the constant multiple law, we have
\[ = \frac{3 \lim_{x \to \infty} 1/x + 4 \lim_{x \to \infty} x^6}{\lim_{x \to \infty} 2 - 7 \lim_{x \to \infty} 1/x^5} \]

Now we need the following theorem: If \( r > 0 \) is a rational number, then \( \lim_{x \to \infty} 1/x^r = 0 \). In the equation above, we note that \( r = 1, 6, 5 \) are all rational numbers, so we get
\[ \lim_{x \to \infty} f(x) = \frac{3 \cdot 0 + 4 \cdot 0}{2 - 7 \cdot 0} = \frac{0}{2} = 0 \]

This concludes our first example.

**Example 2:** Find \( \lim_{x \to -\infty} s(x) \) where \( s(x) = \frac{(x^6-3)^{1/2}}{x^3+x} \).

Convince yourself by the analysis above that we know that the limits of the numerator and denominator exist for all "large negative" numbers (remember...we only care about long-term behavior, not what's going on close to \( x = 0 \).) Then...we want to use the trick that we used above. The highest power in the denominator is \( x^3 \), so we want divide the top and the bottom by \( x^3 \).
\[ \lim_{x \to -\infty} \frac{(x^6-3)^{1/2}}{x^3+x} = \lim_{x \to -\infty} \frac{\frac{1}{x^3}(x^6-3)^{1/2}}{\frac{1}{x^3}(x^3+x)} \]

Now remember, if \( n \) is even, we can't take the \( n \)-the root of a negative number. So, if \( w \) is some number with \( w < 0 \), \( (w^{1/2}) = |w| = -w \). Thus,
\[ (x^6)^{1/2} = (((x^3)^2)^{1/2}) = |x^3| = -x^3 \]
(here we set \( w = x^3 \)). Rearranging, we get \( -(x^6)^{1/2} = x^3 \). Thus,
\[ \lim_{x \to -\infty} \frac{\frac{1}{x^3}(x^6-3)^{1/2}}{\frac{1}{x^3}(x^3+x)} = \lim_{x \to -\infty} \frac{-\frac{1}{x^6/2}(x^6-3)^{1/2}}{\frac{1}{x^3}(x^3+x)} \]

Now we simplify:
\[ \lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \left( -1 \cdot \frac{x^6/x^6 - 3/x^6}{x^3/x^3 + x/x^3} \right) \]

I’ll let you fill in some in between steps, but using the quotient rule and sum rule, we get

\[ = (-1) \left( \lim_{x \to -\infty} (1) - \lim_{x \to -\infty} \frac{3}{x^6} \right)^{1/2} \]

now \( x^6 \) and \( x^2 \) are defined for negative numbers, so we can use our important fact from class today: if \( r > 0 \) is a rational number, and \( x^r \) is defined for all \( x \), then \( \lim_{x \to -\infty} \frac{1}{x^r} = 0 \). Substituting for 0 in the appropriate places we get

\[ \lim_{x \to -\infty} f(x) = (-1)(1)^{1/2} = (-1)(1) = -1 \]

You should try finding \( \lim_{x \to \infty} f(x) \) and the horizontal asymptotes of \( g(x) = \frac{x^8 + 3x^7}{3x^4 + x^2} \). Let me know if you have questions!

**Example 3:** In this example, I want to demonstrate what I mean by *clever multiplication by 1*. Let’s look at the long-term behavior of the function \( h(x) = (16x^2 + x)^{1/2} - 4x \). Above, the highest power of \( x \) controlled the behavior of the function. Here, it’s hard to say what the highest power is - we have to compare \( (16x^2 + x)^{1/2} \) with \( 4x \). So we’re going to algebraically manipulate \( h(x) \) into a more useful form: We know that \( 1 = \frac{(16x^2 + x)^{1/2} + 4x}{(16x^2 + x)^{1/2} + 4x} \) (the number \( (16x^2 + x)^{1/2} + 4x \) is called the **conjugate** of \( (16x^2 + x)^{1/2} - 4x \)).

\[
\begin{align*}
\frac{(16x^2 + x)^{1/2} + 4x}{(16x^2 + x)^{1/2} + 4x} & = \frac{x}{(16x^2 + x)^{1/2} + 4x} \\
\end{align*}
\]

\[
\begin{align*}
\frac{x}{(16x^2 + x)^{1/2} + 4x} & = \frac{x}{(16x^2 + x)^{1/2} + 4x} \\
\end{align*}
\]

So now suppose that we care about \( \lim_{x \to \infty} h(x) \). By the above calculation, we’re looking at

\[ \lim_{x \to \infty} h(x) = \lim_{x \to \infty} \frac{x}{(16x^2 + x)^{1/2} + 4x} \]

Now we can use our first trick, since it’s easy to say that \( \lim_{x \to \infty} (16x^2 + x)^{1/2} + 4x = \infty \) and \( \lim_{x \to \infty} x = \infty \). We multiply the top and the bottom by \( 1/x \), obtaining
\[ \lim_{x \to \infty} h(x) = \lim_{x \to \infty} \frac{(1/x) \cdot x}{(1/x^2)16x^2 + (1/x^2)x^{1/2} + (1/x)4x} \]

\[ = \lim_{x \to \infty} \frac{1}{(16 + 1/x)^{1/2} + 4/x} \]

And an application of our theorem on \(1/x^r\) for \(r > 0\) rational gives

\[ \lim_{x \to \infty} h(x) = \frac{1}{(16)^{1/2}} = \frac{1}{4} \]

To see if you really understand, you should (and I really really mean this!!!) try problems 25, 26, and 27 on page 141 in your book.