Unit 6 – Pythagoras

Sides of Squares

Overview: Participants discover the Pythagorean Theorem inductively by finding the areas of squares.

Objective: **TExES Mathematics Competencies**

II.004.A. The beginning teacher recognizes and extends patterns and relationships in data presented in tables, sequences, or graphs.

III.012.A.The beginning teacher understands axiomatic systems and their components (e.g., undefined terms, defined terms, theorems, examples, counterexamples).

III.013.D. The beginning teacher computes the perimeter, area, and volume of figures and shapes created by subdividing and combining other figures and shapes (e.g., arc length, area of sectors).

III.014.E. The beginning teacher applies concepts and properties of slope, midpoint, parallelism, perpendicularity, and distance to explore properties of geometric figures and solve problems in the coordinate plane.

V.018.A. The beginning teacher understands the nature of proof, including indirect proof, in mathematics.

V.018.C. The beginning teacher uses inductive reasoning to make conjectures and uses deductive methods to evaluate the validity of conjectures.

Geometry TEKS

b.1.A. The student develops an awareness of the structure of a mathematical system, connecting definition, postulates, logical reasoning, and theorems.

b.3.B. The student constructs and justifies statements about geometric figures and their properties.

b.3.C. The student demonstrates what it means to prove mathematically that statements are true.

c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.

d.2.A. The student uses one- and two-dimensional coordinate systems to represent point, lines, line segments, and figures.

d.2.B. The student uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons.

e.1.A. The student finds areas of regular polygons and composite figures.

e.1.C. The student develops, extends, and uses the Pythagorean Theorem.

Background: Participants need the ability to apply the formulas for the area of squares and right triangles.

Materials: centimeter grid paper

New Terms:

Procedures:

Conduct a brief discussion with participants regarding the difficulties embedded in grasping the concept of the Pythagorean Theorem when it is presented as a proven fact without prior conceptual exploration.

Start with the area problem presented in the area unit, which is a proof of the Pythagorean Theorem. Participants return to Unit 5, Applying Area Formulas.

A trapezoid was created by combining two congruent right triangles and an isosceles triangle, as shown. Is the isosceles triangle a right triangle? How do you know? Find the area of the trapezoid two ways: first by using the trapezoid area formula, and then by finding the sum of the areas of the three triangles.

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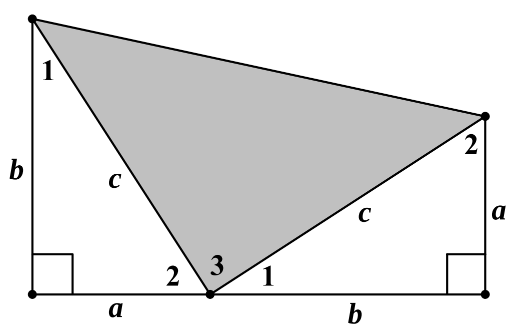


The isosceles triangle is a right triangle because $\angle 1$ and $\angle 2$ are complementary and $\angle 1$, $\angle 2$ and $\angle 3$ form a straight angle, therefore $m\angle 3 = 90°$. If you use the trapezoid area formula, the area of the trapezoid is $\frac{1}{2}(a+b)(a+b)$. If you add the areas of the three triangles, the area of the trapezoid is $\frac{1}{2}c^2 + ab$.

Equating the two and simplifying:

$$\frac{1}{2}(a+b)(a+b) = \frac{1}{2}c^2 + ab.$$

$$\frac{1}{2}(a^2 + 2ab + b^2) = \frac{1}{2}c^2 + ab.$$

$$\frac{1}{2}a^2 + ab + \frac{1}{2}b^2 = \frac{1}{2}c^2 + ab.$$

$$\frac{1}{2}a^2 + \frac{1}{2}b^2 = \frac{1}{2}c^2.$$

$$a^2 + b^2 = c^2.$$

The above proof is attributed to President Garfield. See Pappas (1986).

Discussion questions:
What can be understood conceptually about the Pythagorean Theorem from the proof that emerged from the area problem?
The proof provides no conceptual understanding of the meanings of a^2, b^2, and c^2 as geometric entities.

What van Hiele level is represented by the area problem with respect to the Pythagorean Theorem?
Success with this problem indicates that a person is working at the Deductive Level. The relationship is abstract, derived algebraically, and without connection to the geometric or contextual meanings of a^2, b^2, and c^2.

How is the Pythagorean Theorem usually presented?
In textbooks, often a figure of a right triangle with squares drawn on its sides is presented with the relationship $a^2 + b^2 = c^2$. Students are expected to accept the diagram as a "proof" and then memorize and apply the result. This may be considered to be the final phase of concept development in the van Hiele model. The guided discovery has been omitted.

What concepts must be in place for concept development of the Pythagorean Theorem?
Students must be able to find the side length of a square given its area.

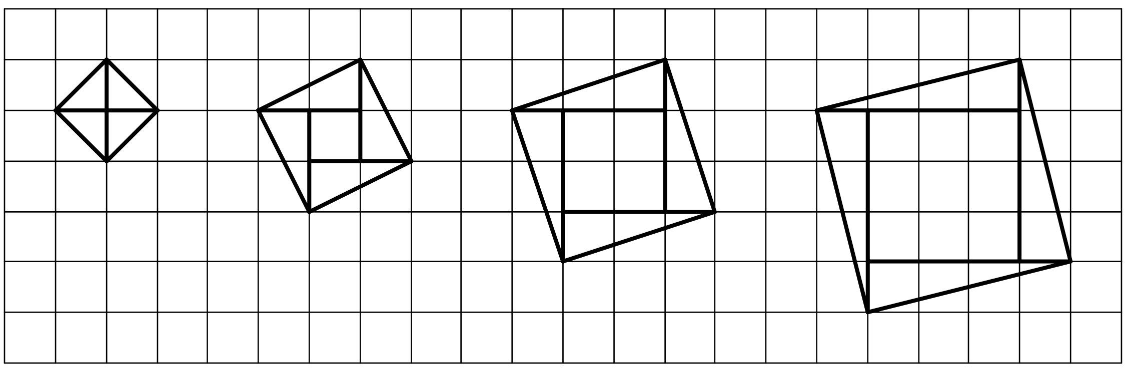
Distribute the activity sheets. Arrange participants in groups of three to complete Part 1 with each participant completing one of the three grids A, B, or C. Participants then complete Part 2 together. Continue working to complete Part 2 prior to whole class discussion.

Part 1

For grids A, B and C, complete the following steps for each line segment. Let the slope of each segment be $\frac{a}{b}$.

- Write the slope, in unsimplified form, next to the segment. In Grid A, $a = 1$; in Grid B, $a = 2$; in Grid C, $a = 3$.
- Using your knowledge of parallel and perpendicular lines, build a square that is on the upper left side of each segment.
- Divide each square into a composite of right triangles and squares by drawing segments in from the vertices, along the horizontal or vertical grid lines.
- Find the area of the original square from the sum of the areas of the composite figures.
- Find the length of each original segment from the area of its square.

Grid A Slope = $\frac{1}{b}$



Slope $= \frac{1}{1}$

Area of square

$= 4 \cdot \frac{1}{2}$

$= 2$

length $= \sqrt{2}$

Slope $= \frac{1}{2}$

Area of square

$= (4 \cdot 1) + 1$

$= 5$

length $= \sqrt{5} \approx 2.24$

Slope $= \frac{1}{3}$

Area of square

$= \left(4 \cdot \frac{3}{2}\right) + 4$

$= 10$

length $= \sqrt{10} \approx 3.16$

Slope $= \frac{1}{4}$

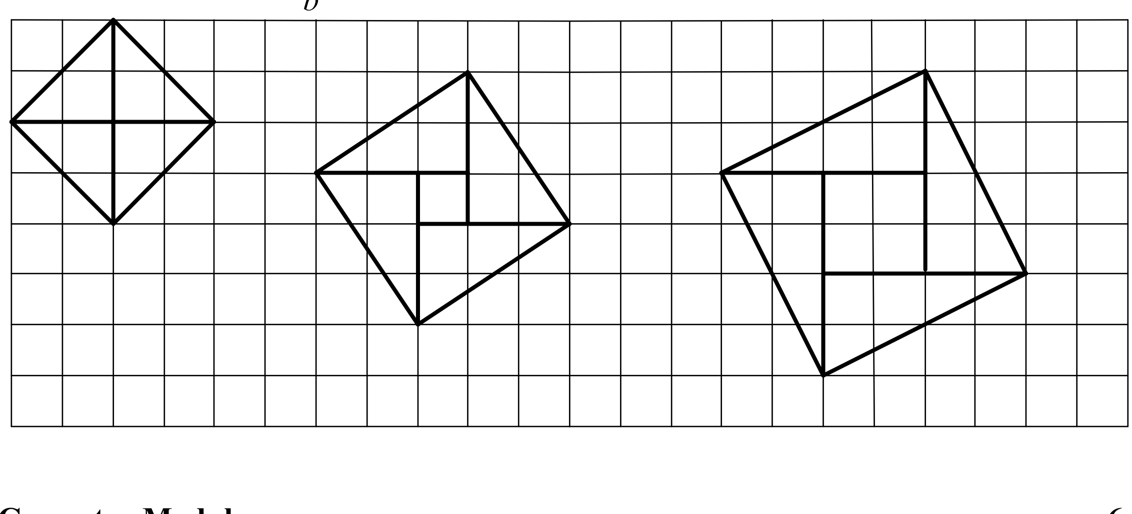
Area of square

$= (4 \cdot 2) + 9$

$= 17$

length $= \sqrt{17} \approx 4.12$

Grid B Slope = $\frac{2}{b}$



$Slope = \frac{2}{2}$

$Area\ of\ square$

$= 4(2)$

$= 8$

$length = \sqrt{8} \approx 2.83$

$Slope = \frac{2}{3}$

$Area\ of\ square$

$= 4(3) + 1$

$= 13$

$length = \sqrt{13} \approx 3.61$

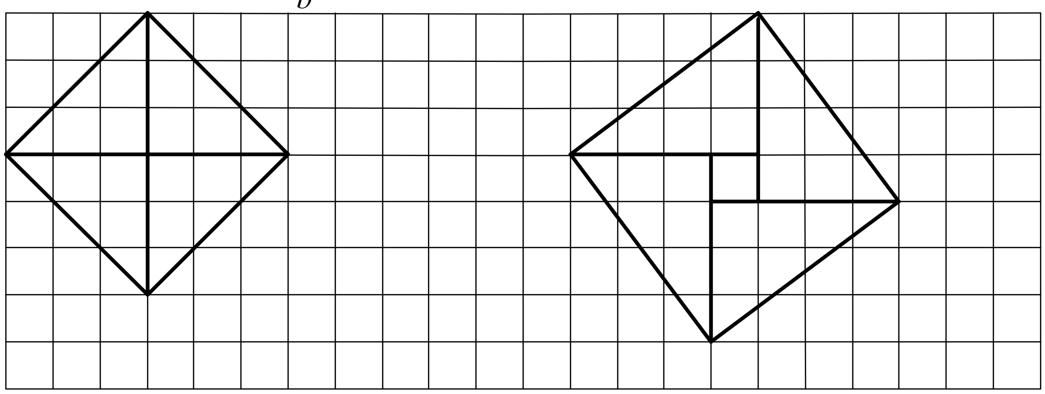
$Slope = \frac{2}{4}$

$Area\ of\ square = 4(4) + 4$

$= 20$

$l\ ength = \sqrt{20} \approx 4.47$

Grid C Slope = $\frac{3}{b}$



$Slope = \frac{3}{3}$

$Area\ of\ square$

$= 4 \cdot \frac{9}{2}$

$= 18$

$length = \sqrt{18} \approx 4.24$

$Slope = \frac{3}{4}$

$Area\ of\ square$

$= (4 \cdot 6) + 1$

$= 25$

$length = \sqrt{25} = 5$

Part 2

Complete the following table. Write the length of the original segment in unsimplified radical form (i.e., $\sqrt{2}$, rather than 1.14…)

	Slope	Area of Original Square	Length of Original Segment
Grid A Slope = $\frac{1}{b}$	$\frac{1}{1}$	2	$\sqrt{2}$
	$\frac{1}{2}$	5	$\sqrt{5}$
	$\frac{1}{3}$	10	$\sqrt{10}$

	$\frac{1}{4}$	17	$\sqrt{17}$
Grid B Slope = $\frac{2}{b}$	$\frac{2}{2}$	8	$\sqrt{8}$
	$\frac{2}{3}$	13	$\sqrt{13}$
	$\frac{2}{4}$	20	$\sqrt{20}$
Grid C Slope = $\frac{3}{b}$	$\frac{3}{3}$	18	$\sqrt{18}$
	$\frac{3}{4}$	25	$\sqrt{25} = 5$

In your group, using the grid figures and the table, discuss and determine relationships among slope numbers, area and segment length. Be prepared to share your findings in whole class discussion.

When most groups have completed Parts 1 and 2, call on a participant to record relationships on the board or easel paper.

The relationships that must emerge are:

- The area of the square is equal to the sum of the squares of the numbers in the unsimplified slope ratio. To generalize, if the segment's slope (in unsimplified form) is $\frac{a}{b}$, then the area of the square on the segment is $a^2 + b^2$.
- The length of the segment is given by $\sqrt{a^2 + b^2}$.

Other relationships that may emerge:

- In the grid figures, when the slopes are of the form $\frac{a}{a}$, there is no middle square, and the area is given by $2a^2$.
- When the numbers in the slope ratio differ by 1, the area of the middle square is 1^2.
- When the numbers in the slope ratio differ by 2, the area of the middle square is 2^2.
- When the numbers in the slope ratio differ by 3, the area of the middle square is 3^2.
- The base and the height of each of the four congruent triangles are always the two numbers in the unsimplified slope ratio.
- To generalize, if slope = $\frac{a}{b}$ (in unsimplified form), the area of the middle square is $(a - b)^2$; the area of each of the four congruent triangles is $\frac{ab}{2}$.

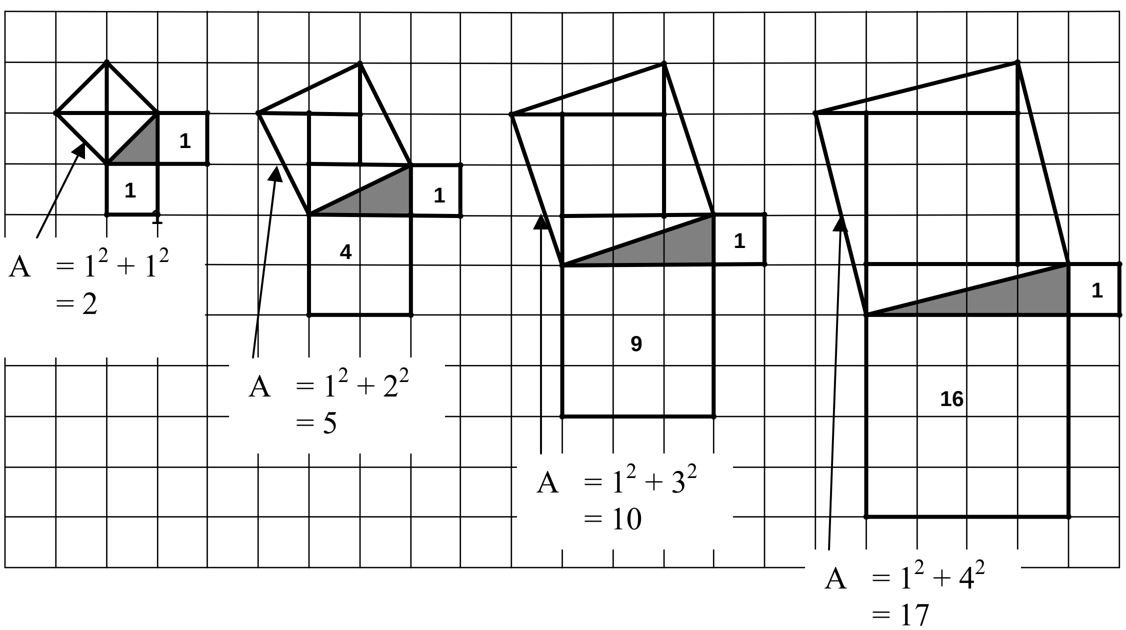
$$\text{Total area} = (a-b)^2 + 4\left(\frac{ab}{2}\right)$$
$$= a^2 - 2ab + b^2 + 2ab$$
$$= a^2 + b^2.$$

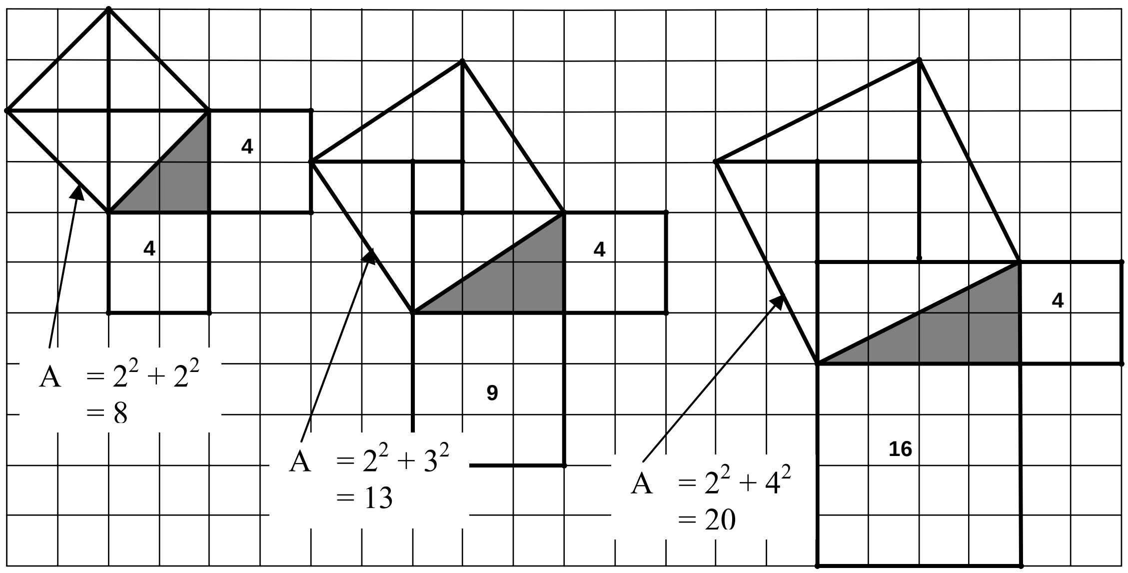
Part 3

Note: The activity sheet does not include Part 3. This part is trainer-directed and makes the connection to the right triangle with squares on the legs and hypotenuse. Using the vertical and horizontal grid lines, draw the right triangle on an original side of one of the squares. Continue the discussion as follows:

What is the geometric meaning of a^2 or b^2?
If a (or b) is a length measurement, then a^2 (or b^2) is the area of a square with side measurement a (or b).

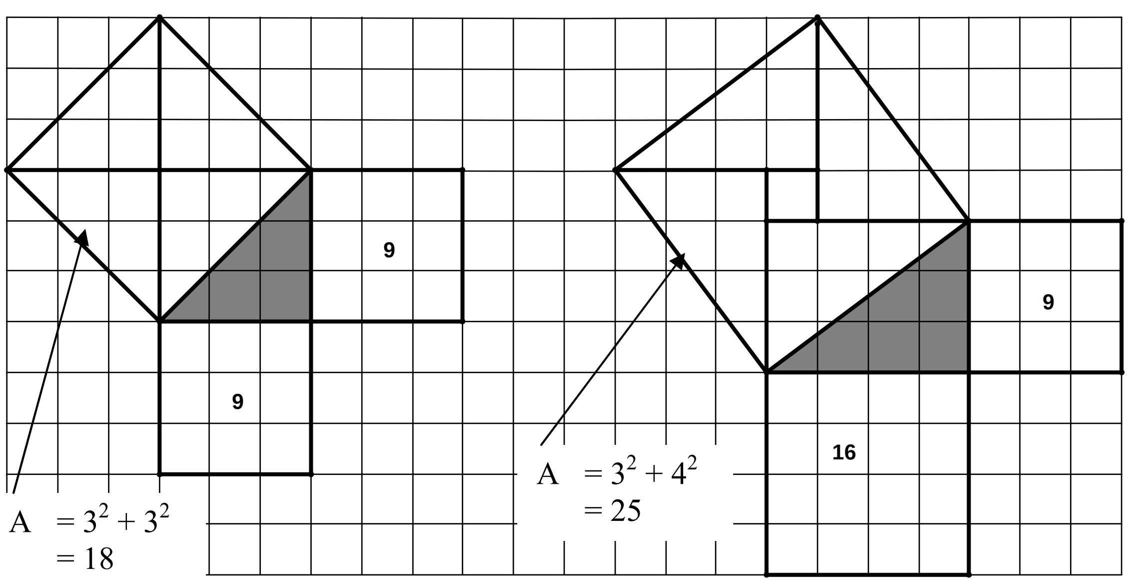
Participants draw and shade in the right triangle on one of the sides of the square using the side of the square as the hypotenuse. Show its legs, which are a and b, as vertical and horizontal segments. Then draw in the squares that represent the areas a^2 and b^2.





When most participants have completed drawing the squares with horizontal and vertical sides, make sure that they also compute the areas and connect the symbolic statement of the Pythagorean Theorem, $a^2 + b^2 = c^2$, to the geometric, pictorial representations in each figure.

Ask the following question and allow groups a few minutes to share their answers.
How would you express the Pythagorean Theorem in words now that you have a symbolic and pictorial representation?
Considering a right triangle with squares constructed on the legs and the hypotenuse, the area of the largest square, the one on the hypotenuse, is equal to the sum of the areas of the squares on the two legs.



Make a connection to triangle congruence using the correspondence of hypotenuse and leg, or hypotenuse and acute angle between right triangles.

Are two right triangles congruent if they have congruent hypotenuses and one of the legs on one triangle is congruent to one of the legs on the other triangle?
If the lengths of the hypotenuse and one leg of a right triangle are known, then, using the Pythagorean Theorem, the other leg is determined. Since the two triangles now have three congruent sides congruent to three congruent sides, SSS, the triangles are congruent. This is referred to as the Hypotenuse-Leg Triangle Congruence Theorem (HL).

Are two right triangles congruent if they have congruent hypotenuses and one of the acute angles on one triangle is congruent to one of the acute angles on the other triangle?
If the measures of two angles are known, namely the right angle and the acute angle, then the third angle is known. Since one triangle has three angles and the hypotenuse congruent to three angles and the hypotenuse on the other triangle, ASA, the triangles must be congruent. This is referred to as the Hypotenuse-Angle Triangle Congruence Theorem (HA).

Success in Part 1 of the activity indicates that participants are working at the van Hiele Descriptive Level because the figures are analyzed in terms of their components. The relationships between the component parts are identified.

In Part 2, success indicates that participants are working at the Relational Level. The activity requires them to formulate and use definitions and to draw conclusions from informal arguments.

In Part 3, success indicates that participants are working at the Relational Level, because algebraic relationships are connected to their geometric counterparts deductively.

In the next activity, participants establish the relationship between the areas of the squares on the two shorter sides and the area on the longest side of an acute and an obtuse triangle.

Sides of Squares

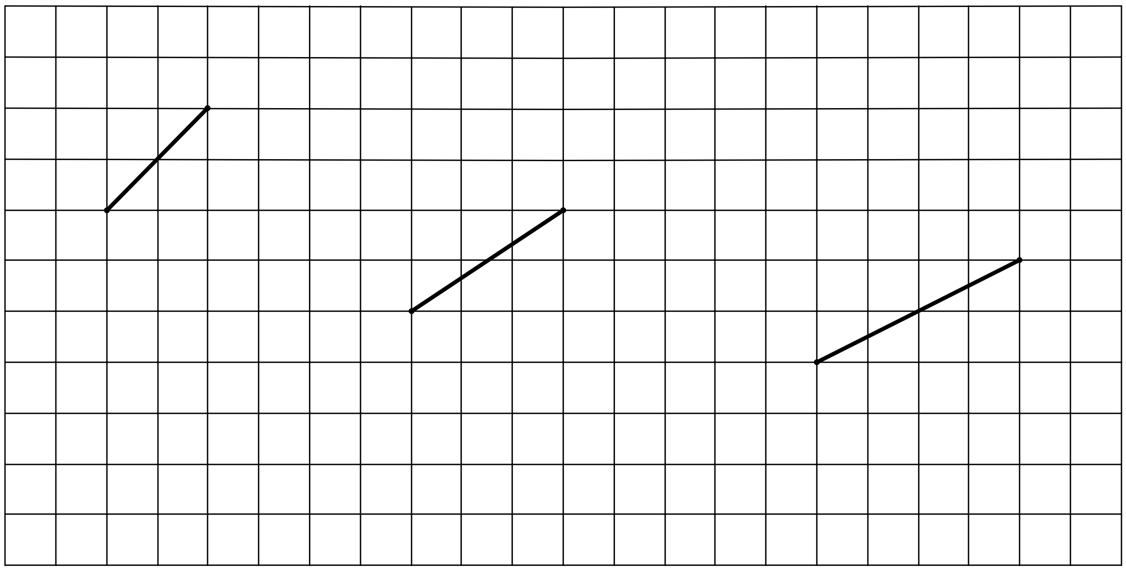
Part 1

For grids A, B and C, complete the following steps for each line segment. Let the slope of each segment be $\frac{a}{b}$.

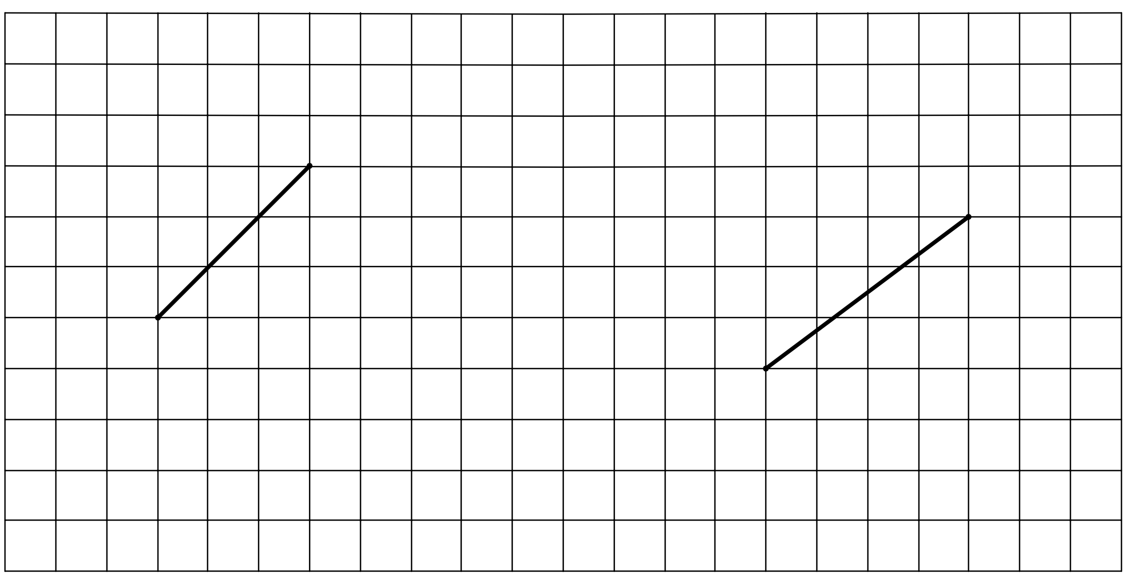
- Write the slope, in unsimplified form, next to the segment. In Grid A, a = 1; in Grid B, a = 2 ; in Grid C, a = 3.
- Using your knowledge of parallel and perpendicular lines, build a square that is on the upper left side of each segment.
- Divide each square into a composite of right triangles and squares by drawing segments in from the vertices, along the horizontal or vertical grid lines.
- Find the area of the original square from the sum of the areas of the composite figures.
- Find the length of each original segment from the area of its square.

Grid A Slope = $\frac{1}{b}$

Grid B Slope = $\frac{2}{b}$



Grid C Slope = $\frac{3}{b}$



Part 2

Complete the following table. Write the length of each of the original segments in unsimplified radical form (i.e., $\sqrt{2}$, rather than 1.14…)

	Slope	**Area of Original Square**	**Length of Original Segment**
Grid A Slope = $\frac{1}{b}$			
Grid B Slope = $\frac{2}{b}$			
Grid C Slope = $\frac{3}{b}$			

In your group, using the grid figures and the table, discuss and determine relationships among slope numbers, area and segment length. Be prepared to share your findings in whole class discussion.

Part 3

Conclusions

Squares on the Sides of Acute or Obtuse Triangles

Overview: Participants test the validity of the Pythagorean Theorem inductively by finding the areas of squares on the sides of acute and obtuse triangles.

Objective: **TExES Mathematics Competencies**
V.018.A. The beginning teacher understands the nature of proof, including indirect proof, in mathematics.
V.018.C. The beginning teacher uses inductive reasoning to make conjectures and uses deductive methods to evaluate the validity of conjectures.
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.

Geometry TEKS
b.3.C. The student demonstrates what it means to prove mathematically that statements are true.
d.2.A. The student uses one- and two-dimensional coordinate systems to represent point, lines, line segments, and figures.
e.1.C. The student develops, extends, and uses the Pythagorean Theorem.

Background: Participants need to be able to find side lengths of squares when given their area.

Materials: centimeter grid paper, centimeter ruler

New Terms:

Procedures:

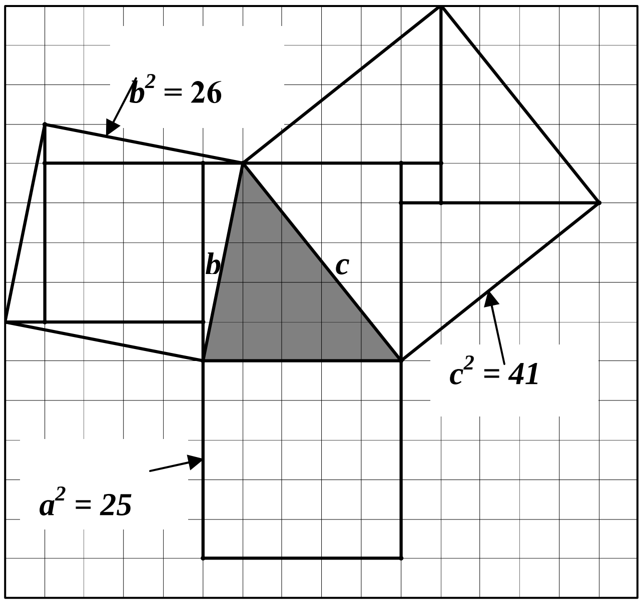
Distribute centimeter grid paper to participants.

1. Draw a scalene acute triangle in the center of the grid paper, with one of its sides along the horizontal or vertical lines. Label the longest side of the triangle *c* and the other two sides *a* and *b*. Make sure each vertex is at the intersection of a horizontal and a vertical grid line.

   Draw the squares on each of the three sides of the triangle. Using the same dividing process as in the previous activity, find the area of each of the squares, a^2, b^2, and c^2.

   Determine the relationship among a^2, b^2, and c^2 for an acute triangle.

An example is shown below.



$a^2 + b^2 = 26 + 25 = 51.$
$c^2 = 41.$
$51 > 41.$
$a^2 + b^2 > c^2.$

If participants find the process confusing, it may help to cover over most of the acute triangle, leaving visible only the segment whose square is being constructed.

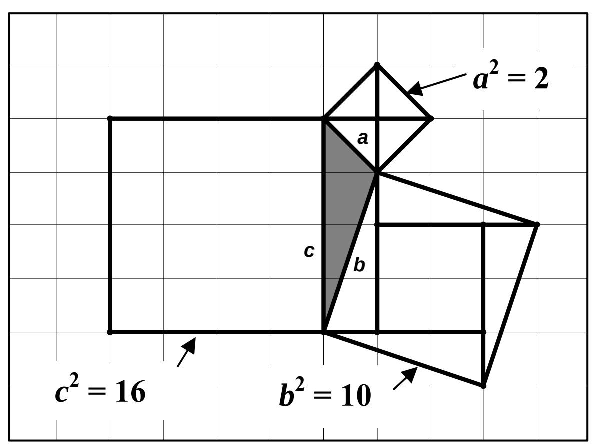
2. Draw a scalene obtuse triangle in the center of the grid paper with one of its sides along the horizontal or vertical lines. Label the longest side *c*, and the other two sides *a* and *b*. Make sure each vertex is at the intersection of a horizontal and a vertical grid line.

   As with the acute triangle, draw the squares on the three sides, and find the areas of the three squares, a^2, b^2, and c^2.

   Determine the relationship among a^2, b^2, and c^2 for an obtuse triangle.

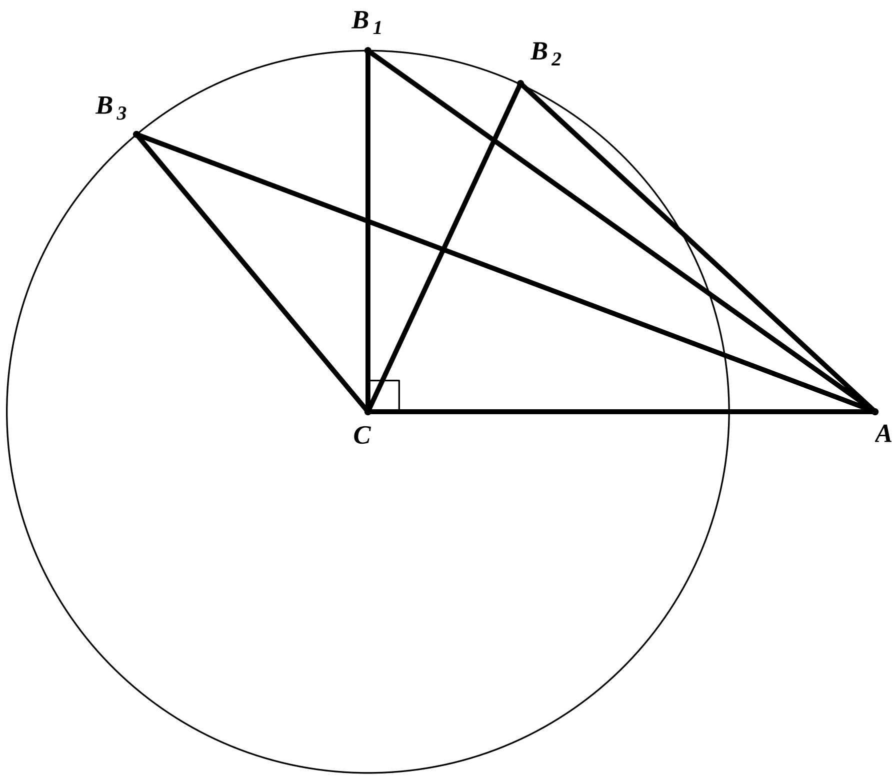
   An example is shown.

   $a^2 + b^2 = 2 + 10 = 12.$
   $c^2 = 16.$
   $12 < 16.$
   $a^2 + b^2 < c^2.$



Does the relationship $a^2 + b^2 = c^2$ still apply to acute triangles?
No. In acute triangles, if c is the longest side, $a^2 + b^2 > c^2$.

Does the relationship $a^2 + b^2 = c^2$ still apply to obtuse triangles?
No. In obtuse triangles, if c is the longest side, $a^2 + b^2 < c^2$.
The following illustration provides additional support for this result. Consider the three triangles, ΔAB_1C, ΔAB_2C, and ΔAB_3C. Two sides in each of the three triangles are congruent. $\overline{AC}$ is common to all three triangles, and $B_1C = B_2C = B_3C$, since all are radii of circle C.



ΔAB_1C is a right triangle, with right angle at the center of circle C.
Therefore, $(AC)^2 + (B_1C)^2 = (B_1A)^2$.

ΔAB_2C is an acute triangle.
Two sides of ΔAB_2C are congruent to two sides of ΔAB_1C ($B_2C = B_1C;\ AC = AC$).
But $B_1A > B_2A$, therefore $(AC)^2 + (B_2C)^2 > (B_2A)^2$.

ΔAB_3C is an obtuse triangle.
Two sides of ΔAB_3C are congruent to two sides of ΔAB_1C ($B_3C = B_1C;\ AC = AC$).
But $B_1A < B_3A$, therefore $(AC)^2 + (B_3C)^2 < (B_3A)^2$.

Success in the activity indicates that participants are working at the Relational Level, because they provide informal, deductive arguments for the relationship of the squares of lengths of sides in a triangle. Moreover, properties of different triangles are interrelated.

Squares on the Sides of Acute or Obtuse Triangles

1. Draw a scalene acute triangle in the center of the grid paper, with one of its sides along the horizontal or vertical lines. Label the longest side of the triangle *c* and the other two sides *a* and *b*. Make sure each vertex is at the intersection of a horizontal and a vertical grid line.

   Draw the squares on each of the three sides of the triangle. Using the same dividing process as in the previous activity, find the area of each of the squares, a^2, b^2, and c^2.

   Determine the relationship among a^2, b^2, and c^2 for an acute triangle.

2. Draw a scalene obtuse triangle in the center of the grid paper with one of its sides along the horizontal or vertical lines. Label the longest side *c*, and the other two sides *a* and *b*. Make sure each vertex is at the intersection of a horizontal and a vertical grid line.

   As with the acute triangle, draw the squares on the three sides, and find the areas of the three squares, a^2, b^2, and c^2.

   Determine the relationship among a^2, b^2, and c^2 for an obtuse triangle.

Applying Pythagoras, Part I

Overview: Participants apply the Pythagorean Theorem to a variety of problems.

Objective: **TExES Mathematics Competencies**
II.006.G. The beginning teacher models and solves problems involving linear and quadratic equations and inequalities using a variety of methods, including technology.
III.011.D. The beginning teacher applies the Pythagorean Theorem, proportional reasoning, and right triangle trigonometry to solve measurement problems.
III.013.D. The beginning teacher computes the perimeter, area, and volume of figures and shapes created by subdividing and combining other figures and shapes (e.g., arc length, area of sectors).
V.019.C. The beginning teacher translates mathematical ideas between verbal and symbolic forms.

Geometry TEKS
b.4. The student uses a variety of representations to describe geometric relationships and solve problems.
c.3. The student identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30 -60 - 90) and triangles whose sides are Pythagorean triples.
e.1.C. The student develops, extends, and uses the Pythagorean Theorem.

Background: Participants should be able to apply the Pythagorean Theorem.

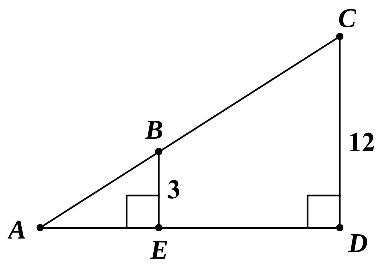
Materials: calculator

New Terms:

Procedures:

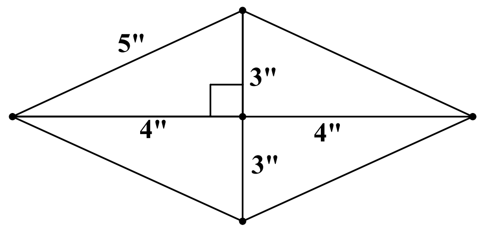
Distribute the activity sheet. Participants work independently or in small groups to complete the six problems.

1. Find the area in square units of trapezoid *BCDE* if the length of $\overline{AC}$ is 20 units, the length of $\overline{DC}$ is 12 units, and the length of $\overline{BE}$ is 3 units.



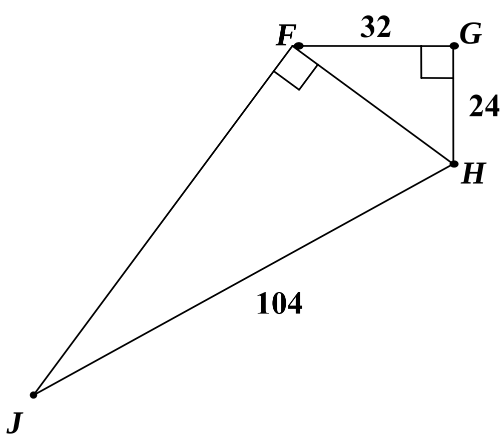
Applying the Pythagorean Theorem on ΔACD, $AD = 16$ units.
$\Delta ABE \sim \Delta ACD$. BE is one-fourth of CD; therefore AE is one-fourth of AD. $AE = 4$ units.
$DE = AD - AE = 12$ units.
Area of trapezoid BCDE
$$= \frac{(3+12)}{2}12 = 90 \text{ square units.}$$

2. The diagonals of a rhombus are 6 inches and 8 inches. What is the perimeter, in inches, of the rhombus?



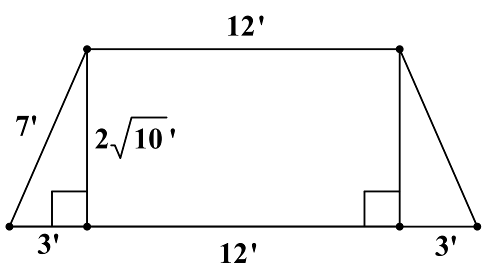
Applying the Pythagorean Theorem, the length of each side of the rhombus is 5 inches.
The perimeter is 20 inches.

3. What is the area in square units of the quadrilateral *FGHJ*?



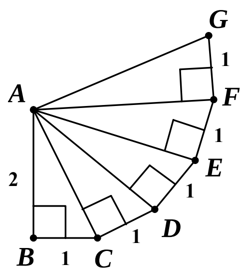
The area of ΔFGH is 344 square units.
Applying the Pythagorean Theorem to ΔFGH, $FH = 40$ units.
Applying the Pythagorean Theorem to ΔFHJ, $FJ = 96$ units.
The area of ΔFHJ is 1920 square units.
The area of quadrilateral FGHJ is 2304 square units.

4. Find the area of the isosceles trapezoid in square feet. Express your answer in simplest radical form.



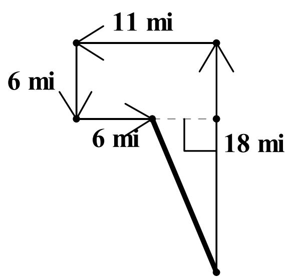
Draw the perpendicular segments from the ends of the short base to the long base, forming right triangles.
Applying the Pythagorean Theorem, the height of the triangles and the height of the trapezoid is $2\sqrt{10}$ ft.
The area of the trapezoid is $30\sqrt{10}\ ft^2$.

5. What is the length of $\overline{AG}$?



Applying the Pythagorean Theorem on ΔABC, $AC = \sqrt{5}$ units.
Applying the Pythagorean Theorem on ΔACD, $AD = \sqrt{6}$ units.
Continuing in this fashion
$AG = \sqrt{9} = 3$ units.

6. A traveler drove 18 miles north, then 11 miles west, then 6 miles south, and then 6 miles east. In miles, how far "as the crow flies" was the traveler from his original starting point?

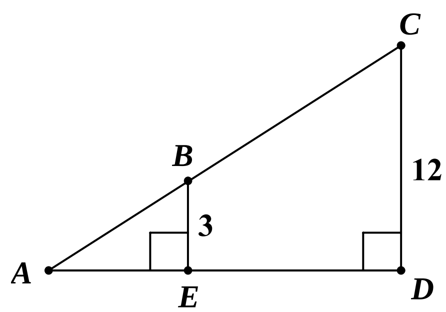


Sketch the route as shown. The legs of the right triangle measure 12 miles and 5 miles. Using the Pythagorean Theorem, the hypotenuse measures 13 miles. The traveler was 13 miles from his original starting point.

Success in this activity indicates that participants are working at the van Hiele Relational Level, because they use properties of different figures along with formulas to solve problems.

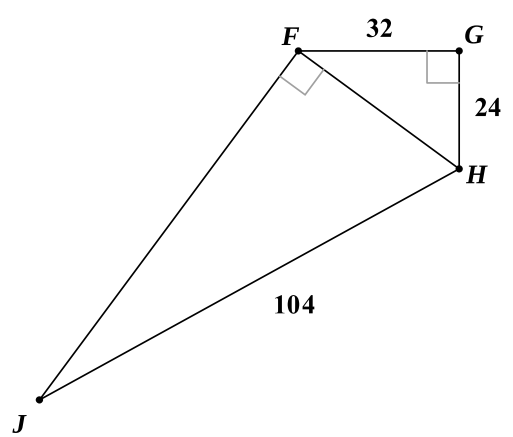
Applying Pythagoras, Part I

1. Find the area in square units of trapezoid *BCDE* if the length of $\overline{AC}$ is 20 units, the length of $\overline{DC}$ is 12 units, and the length of $\overline{BE}$ is 3 units.

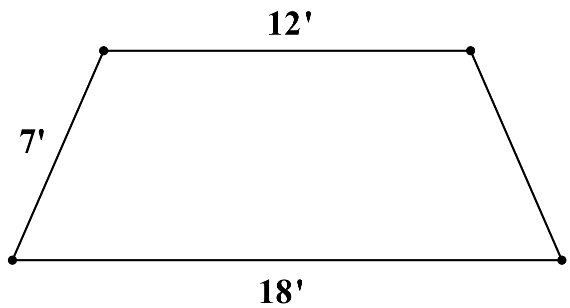


2. The diagonals of a rhombus are 6 inches and 8 inches. What is the perimeter, in inches, of the rhombus?

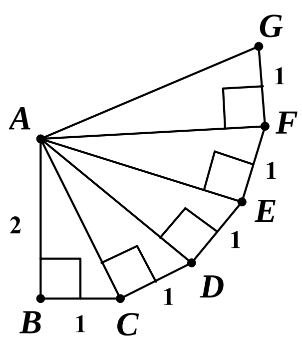
3. What is the area in square units of the quadrilateral *FGHJ*?



4. Find the area of the isosceles trapezoid in square feet. Express your answer in simplest radical form.



5. What is the length of $\overline{AG}$?



6. A traveler drove 18 miles north, then 11 miles west, then 6 miles south, and then 6 miles east. In miles, how far "as the crow flies" was the traveler from his original starting point?

Pythagorean Triples

Overview: Participants discover relationships that will enable them to quickly recall Pythagorean triples.

Objective: **TexES Mathematics Competencies**
II.004.A. The beginning teacher recognizes and extends patterns and relationships in data presented in tables, sequences, or graphs.
II.006.G. The beginning teacher models and solves problems involving linear and quadratic equations and inequalities using a variety of methods, including technology.

Geometry TEKS
b.3.C. The student demonstrates what it means to prove mathematically that statements are true.
c.3. The student identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.
e.1.C. The student develops, extends, and uses the Pythagorean Theorem.
f.1. The student develops, applies, and justifies triangle similarity relationships, such as right triangle ratios, trigonometric ratios, and Pythagorean triples.

Background: Participants should be able to apply the Pythagorean Theorem.

Materials: transparencies of the tables for the activity, calculator

New Terms: Pythagorean triple

Procedures:

Introduce the activity by referring back to the problem solving section of the last activity. Problems 1, 2, 3, and 6 include right triangles whose sides are all natural numbers. When three natural numbers satisfy the Pythagorean Theorem, they are called *Pythagorean triples* or triplets. Remind participants to add the new term Pythagorean triple to their glossaries.

The triples in these problems are in ratios of 3: 4: 5 and 5: 12: 13. Multiples of these triples appear in 1, 2, and 3 of the current activity sheet.

Interesting patterns and generalizations exist in certain groups of triples. The lengths of the legs of a right triangle are a and b, and the length of the hypotenuse is c. In the first table, a is odd; in the second table, a is even.

Participants examine patterns within triples to enable them to easily remember and generate other Pythagorean triples.

Distribute the activity sheet. The activity requires instructor-participant interaction throughout.

Table 1: a is odd.

Work with your group to complete the three rows where $a = 3$, 5, and 7. Be prepared to share in whole class discussion.

Participants complete the row beginning with $a = 3$. The columns labeled a^2, b^2, and c^2 are for checking that the numbers chosen for a, b, and c are indeed Pythagorean triples.

a	***b***			***c***	a^2 +	b^2 =	c^2
3	4			*5*	*9*	*16*	*25*
5	12			*13*	*25*	*144*	*169*
7	*24*			25	*49*	*576*	*625*

Cover the columns for b^2 and c^2.

What patterns can be seen in this part of the table?

a	***b***			***c***	a^2
3	4			*5*	*9*
5	12			*13*	*25*
7	*24*			25	*49*

Several patterns may be discerned, but the following two patterns must emerge:
$b + c = a^2$ and
b and c differ by *1*, or $c - b = 1$.

Using this information, if $a = 9$, then $a^2 = 81$. What are the values for b and c?
$b + c = 81$, and $c - b = 1$.
$b = 40$, $c = 41$.

How may one find b and c quickly?
Find one half of a^2, which is 40.5. Subtract 0.5 from 40.5 to find b, which is 40. Add 0.5 to 40 to find c, which is 41.
Alternately, subtract 1 from a^2 to get 80. Divide 80 by 2 to get b, which is 40. Add 1 to 40 to get c, 41.

Write the following numbers in the table. Participants find the missing numbers using the inductive rule determined above.

a	***b***		***c***	***a²***
3	4		*5*	*9*
5	12		*13*	*25*
7	*24*		25	*49*
9				
11				
	112			
			221	

Do the numbers 9, 40, and 41 form a Pythagorean triple? Test the numbers in the relationship $a^2 + b^2 = c^2$.
The numbers 9, 40, and 41 form a Pythagorean triple, because they satisfy $a^2 + b^2 = c^2$.
$9^2 + 40^2 = 81 + 1600 = 1681 = 41^2$.

What is the triple for $a = 11$? Explain.
The numbers 11, 60, and 61 form a Pythagorean triple.
If $a = 11$, then $a^2 = 121$.
We also know $b + c = 121$.
To find b and c quickly, find half of 121, which is 60.5.
Then subtract 0.5 from 60.5 to get b, which is 60.
Add 0.5 to 60.5 to get c, which is 61.
Note that 60 + 61 = 121.
To check: $11^2 + 60^2 = 121 + 3600 = 3721= 61^2$
Thus, $11^2 + 60^2 = 61^2$, which satisfies the Pythagorean Theorem.

What is the triple for $b = 112$? Explain.
The numbers 15, 112, and 113 form a Pythagorean triple.
If $b = 112$, then $c = 113$.
We know $a^2 = b + c$.
Substituting 112 and 113 for b and c, $a^2 = b + c = 112 + 113 = 225$.
Therefore, $a = 15$.
Check: $a^2 + b^2 = 15^2 + 112^2 = 225 + 12{,}544 = 12{,}769$.
$c^2 = 113^2 = 12{,}769$.
The answers for $a^2 + b^2$ and c^2 are the same, 12,769.

What is the triple for $c = 221$? Explain.
The numbers 21, 220, and 221 form a Pythagorean triple.
If $c = 221$, then $b = 220$.
Since $a^2 = b + c$, $a^2 = 221 + 220 = 441$.
We now know that $a = 21$.
Check: $a^2 + b^2 = 21^2 + 220^2 = 441 + 48{,}400 = 48{,}841$.
$c^2 = 221^2 =48{,}841$.
Thus, $a^2 + b^2$ is the same as c^2. Both are 48,841.

What is the triple in terms of *a*?

a	***b***		***c***	a^2
3	4		5	9
5	12		13	25
7	24		25	49
9	40		41	81
11	60		61	121
15	112		113	225
21	220		221	441
a	$\frac{1}{2}(a^2-1)$		$\frac{1}{2}(a^2+1)$	a^2

Confirm the identity $a^2+\left(\frac{1}{2}(a^2-1)\right)^2=\left(\frac{1}{2}(a^2+1)\right)^2$**.**

$$
\begin{aligned}
&a^2+\left(\frac{1}{2}(a^2-1)\right)^2\\
&=a^2+\frac{1}{4}(a^4-2a^2+1)\\
&=\frac{1}{4}a^4+\frac{1}{2}a^2+\frac{1}{4}\\
&=\left(\frac{1}{2}(a^2+1)\right)^2
\end{aligned}
$$

Insert $a = 1$ in the first row of the table.

What is the triple for *a* = 1, the first odd whole number? Explain.
The triple is 1, 0, 1.
$a^2 = 1$ and
$b + c = 1$, or $c - b = 1$, which results in
$b = 0$ and $c = 1$.
Algebraically, $a = 1$ is valid. However, geometrically there is no triangle with a side length equal to 0.

a	***b***		***c***	a^2
1	0		1	1
3	4		5	9
5	12		13	25

Table 2: *a* is even.

Before participants start Table 2, instruct them to leave the first row blank, and start the second row with *a* is 4.Work with your group to complete the three rows where $a = 4$, 6, and 8. Be prepared to share in whole class discussion.

a	***b***		***c***	a^2 +	b^2 =	c^2
4	*3*		*5*	*16*	*9*	*25*
6	*8*		*10*	*36*	*64*	*100*
8	*15*		*17*	*64*	*225*	*289*

If needed, mention that the first row, 4, 3, 5, is the only row in which *a* is greater than *b*. Encourage participants to look for patterns in the same way as they did for Table 1.

When almost all participants have found the triples for $a = 4, 6, 8$, conduct a whole group discussion about the patterns participants observed, again covering the b^2 and c^2 columns.

Have participants note that *b* and *c* differ by 2, or, $c - b = 2$, then $b + c = \frac{a^2}{2}$.

Usually, participants will divide a^2 by 2, from the process used in Table 1, and then discover that they need to divide by 2 again to find a number to determine *b* and *c*. For example, if $a^2 = 36$, then divide 36 by 2 to get 18. Divide 18 by 2 again to get 9. Then add 1 to 9 to get $c = 10$, and subtract 1 from 9 to get $b = 8$.

Write 9 in the space between 8 and 10. Similarly, write 4 in the space between 3 and 5, and write 16 in the space between 15 and 17.

a	***b***	***Middle #***	***c***	a^2
4	*3*	*4*	*5*	*16*
6	*8*	*9*	*10*	*36*
8	*15*	*16*	*17*	*64*

What operation performed on a^2 produces the number between *b* and *c*?
Divide by 4.

What operation performed on the "middle number" produces *b* and *c*?
Subtract 1 to get *b*; add 1 to get *c*.

Write the following numbers in the table. Participants find the missing values and then justify the answers.

a	*b*	*Middle #*	*c*	a^2
4	*3*	*4*	*5*	*16*
6	*8*	*9*	*10*	*36*
8	*15*	*16*	*17*	*64*
10				
	35			
			50	

For $a = 10$, $a^2 = 100$.
Divide 100 by 4. The "middle number" is 25 and
$b = 24$, $c = 26$.
Check: $a^2 + b^2 = 10^2 + 24^2$
$= 100 + 576$
$= 676$.
$c^2 = 26^2 = 676$.
$a^2 + b^2 = c^2$.
For $b = 35$, $c = 37$.
The "middle number" = 36.
To get a^2, multiply by 4.
$a^2 = 144$.
$a = 12$.
Check: $a^2 + b^2 = 12^2 + 35^2$
$= 144 + 1225$
$=1369$.
$c^2 = 13^2 = 1369$.
$a^2 + b^2 = c^2$.
For $c = 50$, $b = c – 2$.
$b = 48$.
The "middle number" = 49.
To get a^2, multiply by 4.
$a^2 = 196$.
$a = 14$.
Check: $a^2 + b^2 = 14^2 + 48^2$.
$= 196 + 2304$
$= 2500$.
$c^2 = 50^2 = 2500$.
$a^2 + b^2 = c^2$.

a	*b*	*Middle #*	*c*	a^2
4	*3*	*4*	*5*	*16*
6	*8*	*9*	*10*	*36*
8	*15*	*16*	*17*	*64*
10	*24*	*25*	*26*	*100*
12	*35*	*36*	*37*	*144*
14	*48*	*49*	*50*	*196*

Participants may point out that the "middle number" is a sequence of perfect squares equal to $\left(\frac{a}{2}\right)^2$, which simplifies to $\frac{a^2}{4}$.

What is the triple in terms of *a*?

a	*b*	*middle #*	*c*	a^2
a	$\frac{a^2}{4}-1$	$\frac{a^2}{4}$	$\frac{a^2}{4}+1$	a^2

Confirm the identity $\mathbf{a^2+\left(\frac{a^2}{4}-1\right)^2=\left(\frac{a^2}{4}+1\right)^2}$.

$$a^2+\left(\frac{a^2}{4}-1\right)^2$$
$$=a^2+\left(\frac{a^4}{16}-\frac{a^2}{2}+1\right)$$
$$=\frac{a^4}{16}+\frac{a^2}{2}+1$$
$$=\left(\frac{a^2}{4}+1\right)^2$$

The table begins with $a = 4$. The even whole numbers should begin with 2.
Write the value $a = 2$ in the first row of the table. If participants did not leave the first row blank, they can enter the $a = 2$ row above the table headings.

a	*b*	*middle #*	*c*	a^2
2				
4	3	4	5	16

What is the triple for *a* = 2, the first even whole number?
If $a = 2$, then $a^2 = 4$, $b = 0$, and $c = 2$.
Algebraically 2, 0, 2 is a valid triple, but geometrically 0 is not a valid length measurement for the side of a triangle.

a	*b*	*middle #*	*c*	a^2
2	0	1	2	4
4	3	4	5	16

Point out that in Table 1 (*a* is odd), the triples are all primitive. This means that the three numbers have no common factors. However, in the second table (*a* is even), every other row is primitive, while those in between are all multiples of 2. When 2 is factored out, the other factor is one of the triples from Table 1.

	a	*b*	*c*
Primitive	*4*	*3*	*5*
2[3, 4, 5]	*6*	*8*	*10*
Primitive	*8*	*15*	*17*
2[5, 12, 13]	*10*	*24*	*26*
Primitive	*12*	*35*	*37*
2[7, 24, 25]	*14*	*48*	*50*

Explain why multiples of triples always satisfy the Pythagorean Theorem?
In symbolic terms, explain why $a^2 + b^2 = c^2$, is $(ka)^2 + (kb)^2 = (kc)^2$ is always true for any value of k.

$(ka)^2 + (kb)^2$
$= k^2a^2 + k^2b^2$
$= k^2(a^2 + b^2)$
$= k^2(c^2)$
$= k^2c^2$
$= (kc)^2$.

The patterns for "*a* is odd" and "*a* is even" represent a few of the infinite numbers of Pythagorean triples. These also represent a large number of triples that typically appear in high school texts or test problems. Another common triple is 20, 21, 29, which does not fit either of the patterns.

Table 3 is provided for participants for optional further investigation and notes.

Set up the triple, 20, 21, 29, in a comparable table.

a	*b*	*Middle #*	*c*	a^2
20	*21*	*25*	*29*	*400*

Consider the relationships from Tables 1 and 2. What are the comparable relationships here?

In Table 1, $c - b = 1$, and the "middle number" is $\frac{a^2}{2}$.

In Table 2, $c - b = 2$, and the "middle number" is $\frac{a^2}{4}$.

In this problem, $c - b = 8$, and the "middle number" is $\frac{a^2}{16}$.

Since the "middle number" is a perfect square, find other triples that fit the 20, 21, 29 pattern.

a	*b*	*Middle #*	*c*	a^2
20	**21**	**25**	**29**	**400**
		4		
		9		
		16		
		36		
		49		

a	*b*	*middle #*	*c*	a^2	
20	**21**	**25**	**29**	**400**	*Primitive*
8	*0*	**4**	*8*	*64*	*Not a triangle*
12	*5*	**9**	*13*	*144*	*5, 12, 13 in Table 1*
16	*12*	**16**	*20*	*256*	*3[4, 3, 5]*
24	*32*	**36**	*40*	*576*	*8[3, 4, 5]*
28	*45*	**49**	*53*	*784*	*Primitive*

According to Pappas (1991, p. 79), the ancient Greeks discovered:

If *m* is an odd natural number, then $\left(\frac{m^2+1}{2}\right)^2 = \left(\frac{m^2-1}{2}\right)^2 + m^2$.

The patterns in Table 1 satisfy this identity.

Plato's formula, where *m* is a natural number is:
$(m^2 + 1)^2 = (m^2 - 1)^2 + (2m)^2$.
The patterns in Table 2, where $a = 2m$, an even number, satisfy this identity.

Euclid's method:
If *x* and *y* are integers and if $a = x^2 - y^2$, $b = 2xy$, $c = x^2 + y^2$, then *a*, *b*, and *c*, are integers, such that $a^2 + b^2 = c^2$.
You may interchange *a* and *b* in the formula, so that *a* can be odd for some cases of $a = x^2 - y^2$ or *a* can be even ($a = 2xy$).

Further explorations may be conducted by making generalizations for the $c - b$ value and the corresponding divisibility of a^2.

Success in this activity indicates that participants are working at the Relational Level because they use the Pythagorean Theorem to verify the triples. Since algebraic deductive reasoning is used to generalize the triples, they may also be working at the Deductive Level.

Pythagorean Triples

When three whole numbers satisfy the Pythagorean Theorem, they are called Pythagorean triples or triplets. Plato and Euclid derived formulas for the triples. At the end of this activity, you will examine these formulas. Interesting patterns and generalizations exist in certain groups of triples. Let *a* and *b* represent the lengths of the legs of a right triangle, and let *c* represent the length of the hypotenuse. The two tables below are grouped according to the value of *a*. In the first table *a* is odd; in the second table *a* is even.

Table 1: *a* is odd.

Work with your group to complete the three rows where *a* = 3, 5 and 7.
Be prepared to share in whole class discussion.

a	***b***			***c***	a^2 +	b^2 =	c^2
3	4						
5	12						
7				25			

Notes:

Table 2: *a* is even.

Work with your group to complete the three rows where *a* = 4, 6, and 8.
Be prepared to share in whole class discussion.

a	***b***	***middle #***	***c***	a^2	+ b^2	= c^2

Notes:

Table 3:

a	***b***		***c***	a^2 +	b^2 =	c^2

Notes:

Special Right Triangles

Overview: In this activity participants investigate special right triangles.

Objective: **TExES Mathematics Competencies**
II.006.G. The beginning teacher models and solves problems involving linear and quadratic equations and inequalities using a variety of methods, including technology.
III.011.D. The beginning teacher applies the Pythagorean theorem, proportional reasoning, and right triangle trigonometry to solve measurement problems.
III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.

Geometry TEKS
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
b.3.C. The student demonstrates what it means to prove mathematically that statements are true.
c.3. The student identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.
e.1.C. The student develops, extends, and uses the Pythagorean Theorem.
f.3. In a variety of ways, the student develops, applies, and justifies triangle similarity relationships, such as right triangle ratios, trigonometric ratios, and Pythagorean triples.

Background: Participants need to have knowledge of the Pythagorean Theorem.

Materials: geoboard or geoboard dot paper (provided in the Appendix), unlined 8.5 in. by 11 in. paper

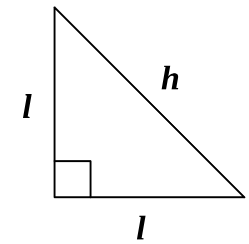
New Terms:

Procedures:

This activity is taken from *Discovering Geometry*: *An Investigative Approach,* 3rd Edition, © 2003, p. 475 with permission from Key Curriculum Press.

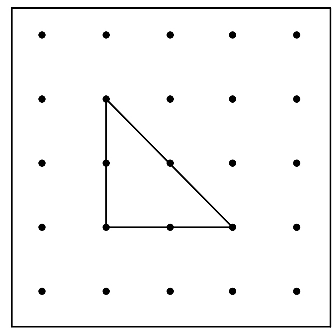
Part 1

On your paper, sketch an isosceles right triangle.
Label the legs *l* and the hypotenuse *h*.



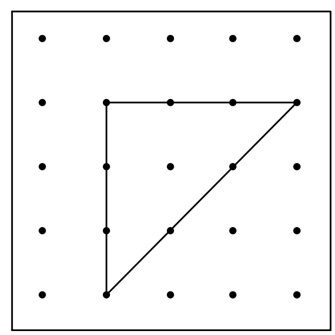
1. Pick any positive integer for *l,* the length of the legs of an isosceles right triangle. Show this triangle on a geoboard or on geoboard dot paper. Use the Pythagorean Theorem to find *h*. Simplify the square root.

$$l^2 + l^2 = h^2$$
$$2^2 + 2^2 = h^2$$
$$4 + 4 = h^2$$
$$8 = h^2$$
$$\sqrt{8} = h$$
$$2\sqrt{2} = h$$



2. Repeat 1 with several other values for *l.* Share results with your group. Do you see a pattern in the relationship between *l* and *h*?

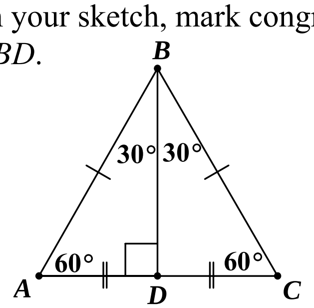
$$l^2 + l^2 = h^2$$
$$3^2 + 3^2 = h^2$$
$$9 + 9 = h^2$$
$$18 = h^2$$
$$\sqrt{18} = h$$
$$3\sqrt{2} = h$$



3. State your conclusion in words and using a drawing.
In a 45°-45°-90° right triangle, the length of the hypotenuse is $\sqrt{2}$ *times the length of the leg.*

Part 2

On your paper, sketch an equilateral triangle, ΔABC. Draw the altitude $\overline{BD}$ from vertex *B* to side $\overline{AC}$. On your sketch, mark congruent sides and show the measures of all angles in ΔABD and ΔCBD.

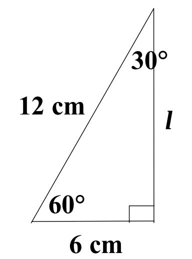


4. What is the relationship between $\overline{AB}$ and $\overline{AD}$? Will this relationship always exist in a 30^o-60^o-90^o triangle? Explain.

   Since AC = AB, then AD = $\frac{1}{2}AB$.

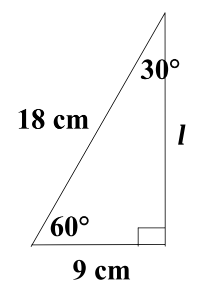
5. Sketch a 30^o-60^o-90^o triangle. Choose any positive integer for the length of the shorter leg. Use the relationship from 4 together with the Pythagorean Theorem to find the length of the other leg. Simplify the square root.

$$6^2 + l^2 = 12^2$$
$$36 + l^2 = 144$$
$$l^2 = 108$$
$$l = 6\sqrt{3}$$



6. Repeat 5 with several values for the length of the shorter leg. Share results with the group.

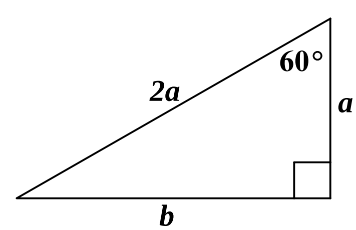
$$9^2 + l^2 = 18^2$$
$$81 + l^2 = 324$$
$$l^2 = 243$$
$$l = 9\sqrt{3}$$



7. Write a generalization for relating the sides of any 30^o-60^o-90^o triangle.
   In a 30^o-60^o-90^o *right triangle, the length of the hypotenuse is twice the length of the short leg. The length of the long leg is* $\sqrt{3}$ *times the length of the short leg.*

8. Use algebra to verify the relationship for any 30^o-60^o-90^o triangle by using the triangle to the right.

$$(2a)^2 = a^2 + b^2$$
$$4a^2 = a^2 + b^2$$
$$3a^2 = b^2$$
$$a\sqrt{3} = b$$



Success in this activity indicates that participants are working at the Relational Level because they use both inductive and deductive methods to generalize properties for special right triangles.

Special Right Triangles

Part 1

On your paper, sketch an isosceles right triangle. Label the legs *l* and the hypotenuse *h*.

1. Pick any positive integer for *l,* the length of the legs of an isosceles right triangle. Show this triangle on a geoboard or on geoboard dot paper. Use the Pythagorean Theorem to find *h*. Simplify the square root.

2. Repeat 1 with several other values for *l.* Share results with your group. Do you see a pattern in the relationship between *l* and *h*?

3. State your conclusion in words and using a drawing.

Part 2

On your paper, sketch an equilateral triangle, ΔABC. Draw the altitude $\overline{BD}$ from vertex *B* to side $\overline{AC}$. On your sketch, mark congruent sides and show the measures of all angles in ΔABD and ΔCBD.

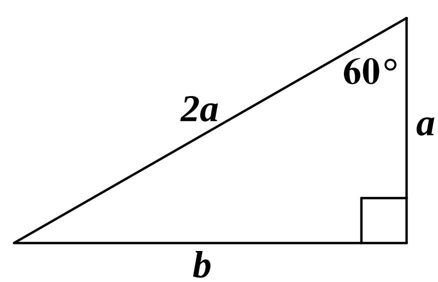
4. What is the relationship between $\overline{AB}$ and $\overline{AD}$? Will this relationship always exist in a 30^o-60^o-90^o triangle? Explain.

5. Sketch a 30^o-60^o-90^o triangle. Choose any positive integer for the length of the shorter leg. Use the relationship from 4 together with the Pythagorean Theorem to find the length of the other leg. Simplify the square root.

6. Repeat 5 with several values for the length of the shorter leg. Share results with the group.

7. Write a generalization relating the sides of any 30^o-60^o-90^o triangle.

8. Use algebra to verify the relationship for any 30^o-60^o-90^o triangle by using the triangle to the right.



Distance Formula

Overview: In this activity participants investigate the length of line segments using the Pythagorean Theorem.

Objective: **TExES Mathematics Competencies**
III.011.D. The beginning teacher applies the Pythagorean theorem, proportional reasoning, and right triangle trigonometry to solve measurement problems.
III.014.E. The beginning teacher applies concepts and properties of slope, midpoint, parallelism, perpendicularity, and distance to explore properties of geometric figures and solve problems in the coordinate plane.

Geometry TEKS
d.2.A. The student uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures.
d.2.C. The student develops and uses formulas including distance and midpoint.
e.1.C. The student develops, extends, and uses the Pythagorean Theorem.

Background: Participants need to be able to understand and use the Pythagorean Theorem.

Materials: centimeter grid paper, centimeter grid transparency, 3 in. square adhesive notes in two colors (one of each color per participant)

New Terms: distance formula

Procedures:

Participants find the length of a non-vertical and non-horizontal line segment whose endpoints lie on a given coordinate plane. A right triangle is drawn using the original line segment as the hypotenuse. Horizontal and vertical segments drawn along the gridlines from the endpoints of the original segment form the legs. The lengths of the legs can be determined by observation. The length of the original segment can be found using the Pythagorean Theorem.

When the segment has endpoints which do not fit on a given coordinate plane, participants develop and apply the distance formula, based on an understanding of the Pythagorean Theorem.

Prior to the training, write an ordered pair on an adhesive note for each participant. On one color adhesive note write an ordered pair whose x- and y-values vary from -8 to 8. On the other color adhesive note, write an ordered pair whose x- and y-values lie between 10 and 50 or between -10 and -50. Try to provide a wide variety of ordered pair choices for the whole class.

Part 1

Begin the activity by distributing an adhesive note in each color to each participant. Each group of four participants will then have four ordered pairs. On centimeter grid paper, using the coordinates between -8 and 8, ask participants to create four line segments using the ordered pairs as endpoints. It is possible to create six segments. Each participant will find the distances between the endpoints.

Lead a whole class discussion in which participants summarize how they found the distance. Possible mathematical relationships to highlight are:

- For non-vertical or non-horizontal lines, draw a vertical line segment and horizontal line segment along the grid lines to form a right triangle. Use the Pythagorean Theorem to find the length of the hypotenuse.
- To find the lengths of the vertical and horizontal legs, count the grid units from endpoint to endpoint.
- Irrational numbers should be simplified, e.g., $\sqrt{24}$ is written $2\sqrt{6}$.

Part 2

Ask participants to repeat the activity using the ordered pairs on the other color adhesive note, having coordinate values greater than 50 or less than -50. Give participants time to grapple with the fact that the grid paper is too small to plot the points unless the scale is changed.

Ask participants to write an expression for the length of the horizontal leg using the *x*-coordinates. Then write a similar expression for the length of the vertical leg using the *y*-coordinates. Then apply the leg measurements to the Pythagorean Theorem to find the distance between the points. Use the expressions to write a general formula for distance.

Lead a whole class discussion in which participants summarize how the distance was found. Ask a participant to describe how his/her group developed the expressions, using the group's examples from Part 2.

Given the coordinates of the endpoints of a line segment, $\left(x_1, y_1\right)$ and $\left(x_2, y_2\right)$, the length of the line segment is given by $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. This is commonly known as the *distance formula*. Remind participants to add the term distance formula to their glossaries.

Success with this activity indicates that participants are performing at the Relational Level since they interrelate algebraic and geometric representations.

Applying Pythagoras, Part II

Overview: Participants apply the Pythagorean Theorem at various levels of challenge.

Objective: **TExES Mathematics Competencies**
III.011.B. The beginning teacher applies formulas for perimeter, area, surface area, and volume of geometric figures and shapes (e.g., polygons, pyramids, prisms, cylinders, cones, spheres) to solve problems.
III.011.D. The beginning teacher applies the Pythagorean theorem, proportional reasoning, and right triangle trigonometry to solve measurement problems.
III.013.D. The beginning teacher computes the perimeter, area, and volume of figures and shapes created by subdividing and combining other figures and shapes (e.g., arc length, area of sectors).
III.018.E. The beginning teacher understands the problem-solving process (i.e., recognizing that a mathematical problem can be solved in a variety of ways, selecting an appropriate strategy, evaluating the reasonableness of a solution).
III.019.E. The beginning teacher understands the use of visual media, such as graphs, tables, diagrams, and animations, to communicate mathematical information.

Geometry TEKS
b.4. The student selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.
c.1. The student identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30 -60 - 90) and triangles whose sides are Pythagorean triples.
d.2.C. The student develops and uses formulas including distance and midpoint.
e.1.A. The student finds areas of regular polygons and composite figures.
e.1.C. The student develops, extends, and uses the Pythagorean Theorem.
f.3. In a variety of ways, the student develops, applies, and justifies triangle similarity relationships, such as right triangle ratios, trigonometric ratios, and Pythagorean triples.

Background: Participants need the ability to apply the Pythagorean Theorem in problems that integrate properties of other planar figures such as triangles, quadrilaterals, and to some extent circles.

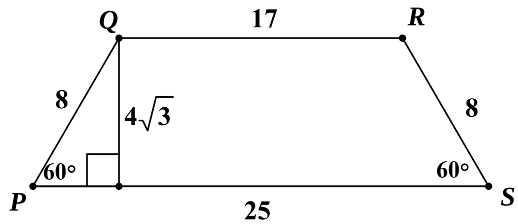
Materials: calculator

New Terms:

Procedures:

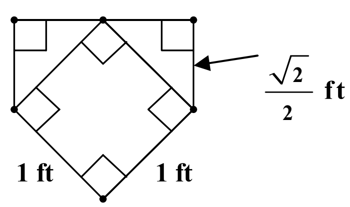
Participants work independently or in pairs. The problems may be assigned for homework.

1. What is the area of trapezoid *PQRS*, whose measures are shown in the diagram?



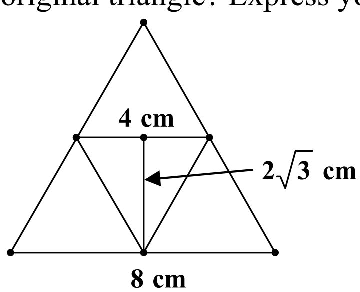
Draw the perpendicular segment from Q or R to $\overline{PS}$. The height of the trapezoid is the same as the height of the 30^o-60^o-90^o triangle: $4\sqrt{3}$ units. The area of trapezoid PQRS is $84\sqrt{3}$ square units.

2. The home plate used in baseball can be produced by adding two isosceles right triangles to a square as shown. What is the area of home plate in square feet?



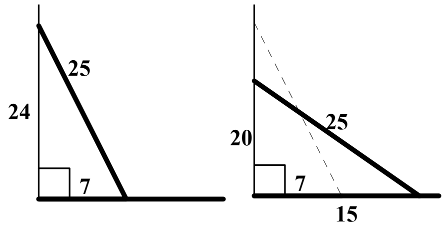
The hypotenuses of the isosceles right triangles measure 1 ft. Using the 1: 1: $\sqrt{2}$ ratio, the legs of the triangles measure $\frac{\sqrt{2}}{2}$ ft. Therefore the area of each triangle is $\frac{1}{2}\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}=\frac{1}{4}ft^2$. The area of the home plate is the sum of the areas of the 1 ft by 1 ft square and the two isosceles right triangles. Therefore, the area is 1.5 ft^2.

3. An equilateral triangle has a perimeter of 24 cm. What is the area in square centimeters of the triangle formed by connecting the midpoints of the sides of the original triangle? Express your answer in simplest radical form.



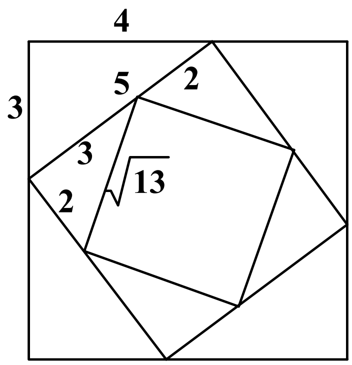
Draw an equilateral triangle and the triangle formed by connecting the midpoints of its sides. Draw the altitude. Use the 1: $\sqrt{3}$:2 ratio for a 30^o-60^o-90^o triangle to find the length of the altitude, $2\sqrt{3}$ cm. The area of the small triangle is $\frac{1}{2}\cdot 2\sqrt{3}\cdot 4 = 4\sqrt{3}$ cm^2.

4. A 25-foot support beam leans against a wall as shown. The base of the beam is 7 feet from the wall. If the top of the beam is lowered 4 feet, how many feet farther away from the wall will the base of the beam be after it is lowered?



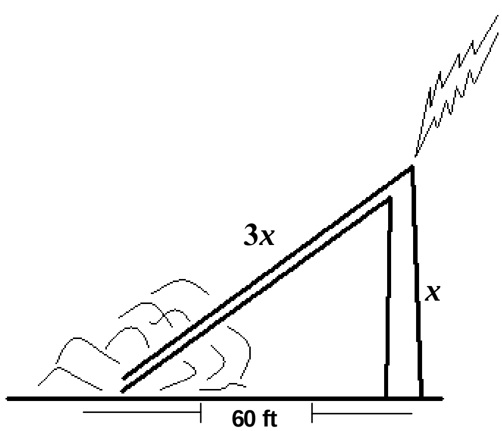
As shown in the diagram, the beam forms a 7:24:25 triangle. When it is lowered the beam forms a 15:20:25 triangle. The base of the beam has moved 8 feet farther away from the wall.

5. The diagram consists of three nested squares. Find the ratio of the area of the smallest square to the area of the largest square.



The side length of the middle square is equal to the length of the hypotenuse of the right triangle with legs of length 3 and 4 units. The side length of the middle square is 5 units.
The side length of the smallest square is equal to the length of the hypotenuse of a right triangle with legs of length 2 and 3 units, which is $\sqrt{13}$.
The ratio of the area of the smallest square to the area of the largest square is $\frac{13}{49}$.

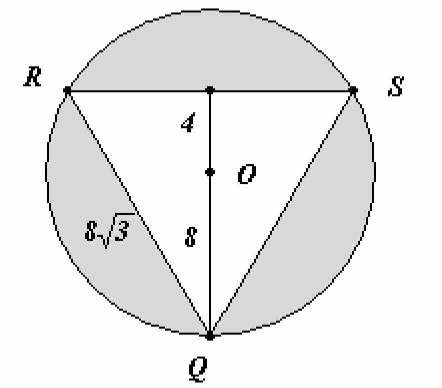
6. Lightning hit a tree one-fourth of the distance up the trunk from the ground, breaking the tree so that its top landed at a point 60 feet from its base, as shown. How many



Let the height of the upright portion of the tree be x, then the hypotenuse is 3x. Apply the Pythagorean Theorem, using leg lengths x and 60, and hypotenuse length 3x.
$x^2 + 60^2 = (3x)^2$. *Solve for x. The original height of the tree was 4x, approximately 84.8 feet.*

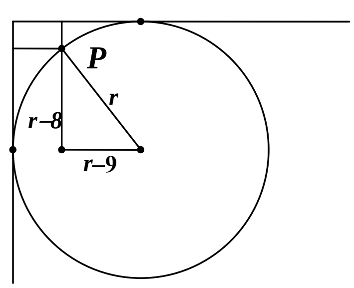
7. Find the perimeter of the polygon shown. The triangles are all right triangles. See the activity sheet for a clear picture of the figure.
The figure is comprised of a series of 30º-60º-90º triangles. The hypotenuse of the largest triangle measures 64 units. The long leg of the second largest triangle measures $16\sqrt{3}$ *units. The long legs of the third, fourth, fifth and sixth triangles measure* $8\sqrt{3}$, $4\sqrt{3}$, $2\sqrt{3}$, *and* $\sqrt{3}$ *units. The perimeter is* $63 + 63\sqrt{3}$, *the sum of the lengths of the long legs of the triangles and the 63 units remaining on the hypotenuse of the largest triangle.*

8. Equilateral triangle *QRS* is inscribed in circle *O*. The radius of the circle is 8 units. The area of the shaded region can be expressed as $x\pi - y\sqrt{3}$, where *x* and *y* are positive integers. Find the ordered pair (*x, y*).



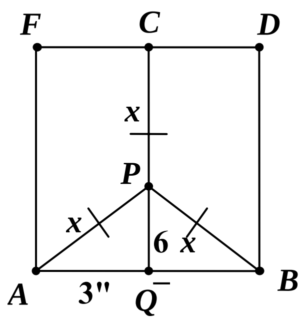
Draw an altitude of the triangle. The center of the circle O divides the altitude into lengths that are in ratio of 2:1. The length of the radius is 8 units. The length of the altitude is 12 units and the length of a side of the triangle is $8\sqrt{3}$ *units.*
The area of the shaded region is
$8^2 \cdot \pi - \frac{1}{2} \cdot 12 \cdot 8\sqrt{3}$ *or* $64\pi - 48\sqrt{3}$ *square units.*
The ordered pair is (*64, 48*).

9. A circular table is pushed into the corner of a square room so that a point *P* on the edge of the table is 8 inches from one wall and 9 inches from the other wall as shown. Find the radius of the circular table in inches.



Draw the radius to P, and the horizontal and vertical legs of the right triangle. The radius is the hypotenuse. The legs are shorter than the radius by 8 and 9 inches respectively. Use the Pythagorean Theorem, $(r-9)^2 + (r-8)^2 = r^2$.
Simplifying: $r^2 - 34r + 145 = 0$ *or* $(r-29)(r-5) = 0$. *This equation has two solutions r = 29 and r = 5. Only r = 29 makes sense in the context of the problem so the radius of the table has a length of 29 inches.*

10. A square with 6" sides is shown. If P is a point in the interior of the square such that the segments $\overline{PA}$, $\overline{PB}$, and $\overline{PC}$ are equal in length, and $\overline{PC}$ is perpendicular to $\overline{FD}$, what is the area, in square inches, of ΔAPB?



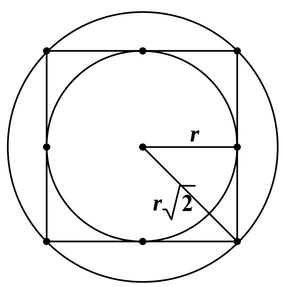
Let CP = x.
Extend $\overline{CP}$ to intersect $\overline{AB}$ at Q.
Altitude PQ = 6 – x.
AP = x. AQ = 3.
Apply the Pythagorean Theorem to the measures of ΔAPQ to solve for x.

$x = 3\frac{3}{4}$ *or* $\frac{15}{4}$ *inches.*

Therefore the area of

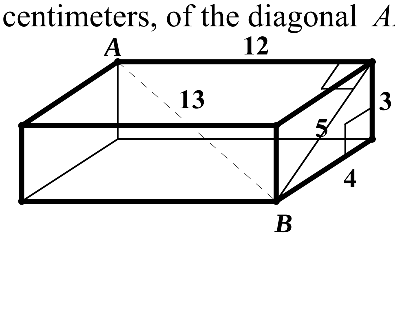
$$\Delta APB = \frac{1}{2}\cdot 3\cdot(6-\frac{15}{4})$$
$$=\frac{1}{2}\cdot 3\cdot\frac{9}{4}=\frac{27}{8}\,in^2.$$

11. The square shown is externally tangent to the smaller circle and internally tangent to the larger circle. Find the ratio of the area of the smaller circle to the area of the larger circle.



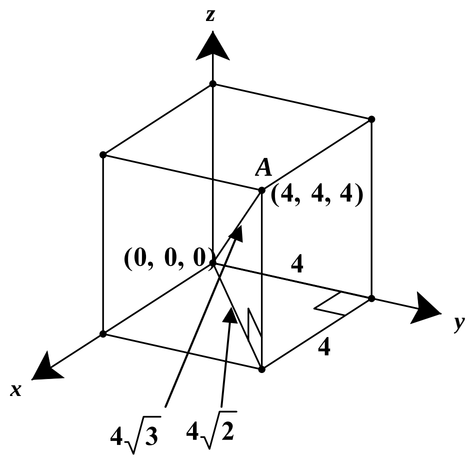
Draw the radius, r, of the small circle to the midpoint of a side of the square. Draw the radius of the large circle to a vertex of the square, forming a 45°-45°-90° right triangle, therefore this radius has a measure of $r\sqrt{2}$. The ratio of the areas of the smaller circle to the larger circle is $\frac{\pi\cdot r^2}{\pi\left(r\sqrt{2}\right)^2}\ \frac{\pi r^2}{\pi 2r^2}=\frac{1}{2}$.

12. A rectangular solid has dimensions 3 cm by 4 cm by 12 cm. What is the length, in centimeters, of the diagonal $\overline{AB}$ of the solid?



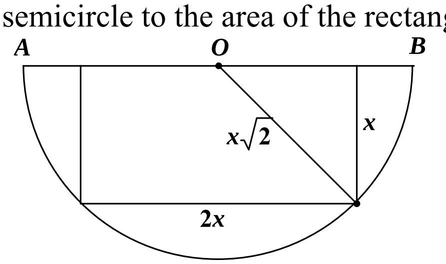
Draw a face diagonal along one of the 3 cm by 4 cm faces. The face diagonal measures 5 cm. The face diagonal, the 12 cm edge, and $\overline{AB}$ are the sides of a right triangle. AB = 13 cm.

13. Point A is located in 3-dimensional space at coordinate position (4, 4, 4). Find the distance from point A to the origin.



The distance from A to the origin is $\sqrt{4^2+4^2+4^2} = 4\sqrt{3}$ *units.*

14. The semicircle shown has center O and diameter $\overline{AB}$. A rectangle whose length is twice its width has two vertices on the semicircle. Find the ratio of the area of the semicircle to the area of the rectangle.

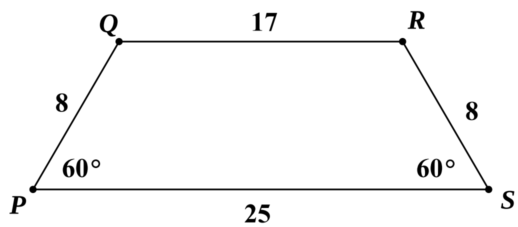


Draw the radius to a vertex of the rectangle on the semicircle. Let the width of the rectangle be x units. The ratio of the area of the semicircle to the area of the rectangle is $\dfrac{\pi x^2}{2x^2} = \dfrac{\pi}{2}$.

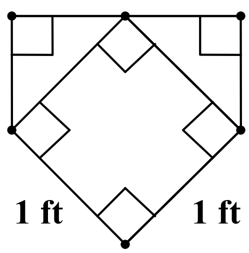
Success with this activity indicated that participants are working at the van Hiele Relational Level, because they use properties and formulas of various figures to solve the problems.

Applying Pythagoras, Part II

1. What is the area of trapezoid *PQRS,* whose measures are shown in the diagram?

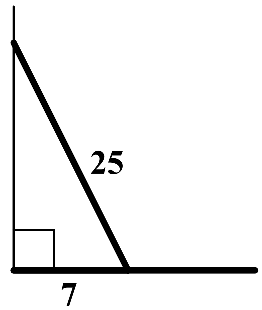


2. The home plate used in baseball can be produced by adding two isosceles right triangles to a square as shown. What is the area of home plate in square feet?

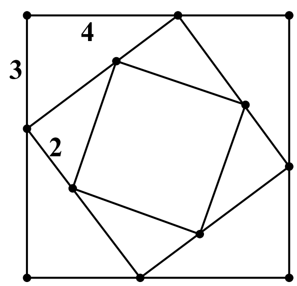


3. An equilateral triangle has a perimeter of 24 cm. What is the area in square centimeters of the triangle formed by connecting the midpoints of the sides of the original triangle? Express your answer in simplest radical form.

4. A 25-foot support beam leans against a wall as shown. The base of the beam is 7 feet from the wall. If the top of the beam is lowered 4 feet, how many feet farther away from the wall will the base of the beam be after it is lowered?



5. The diagram consists of three nested squares. Find the ratio of the area of the smallest square to the area of the largest square.



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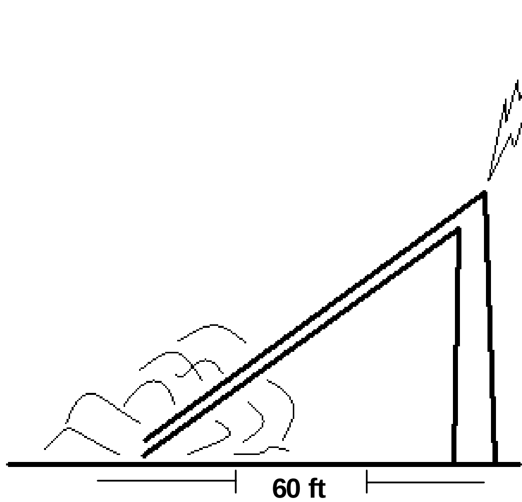
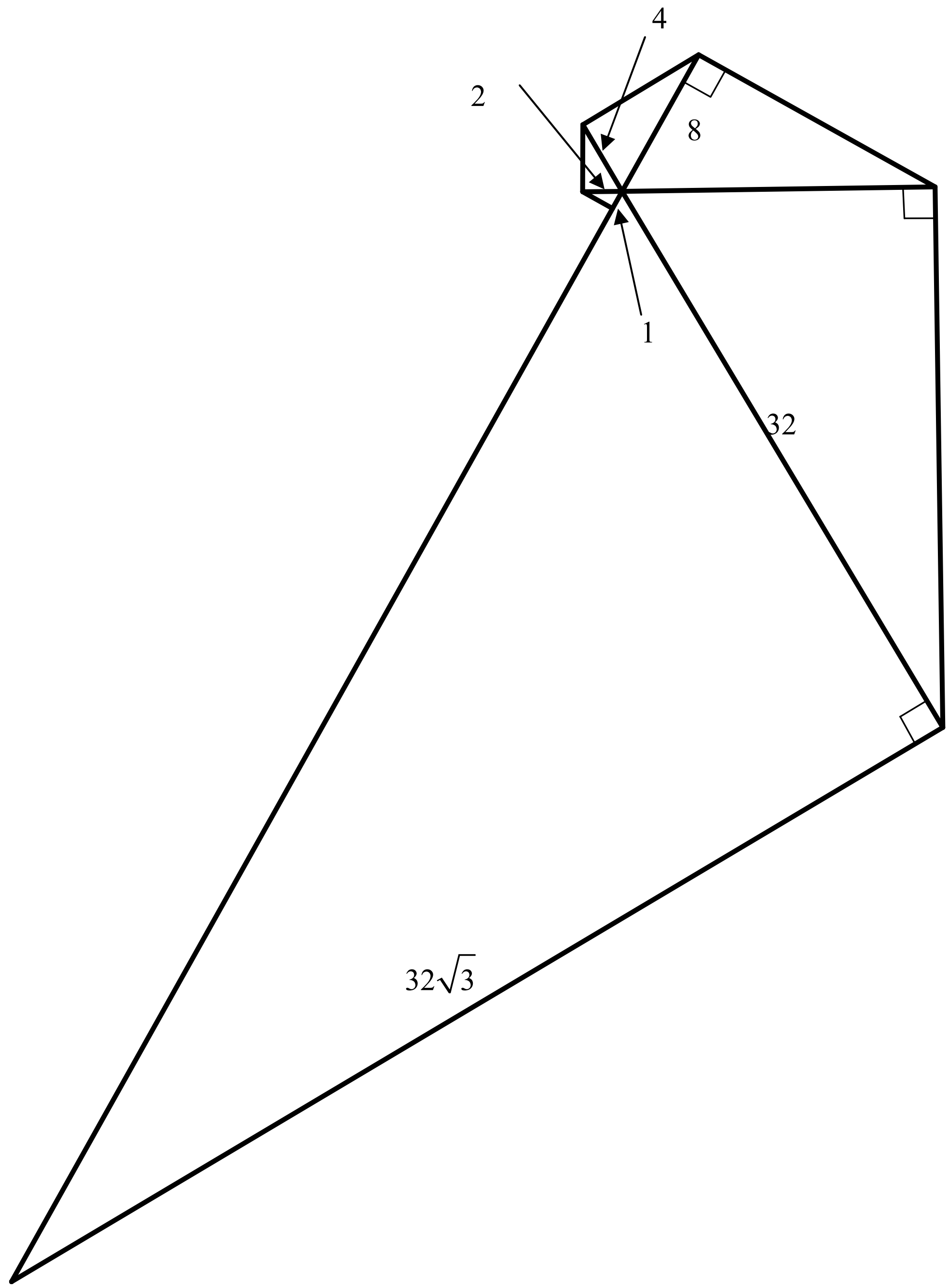


7. Find the perimeter of the polygon shown. The triangles are all right triangles.



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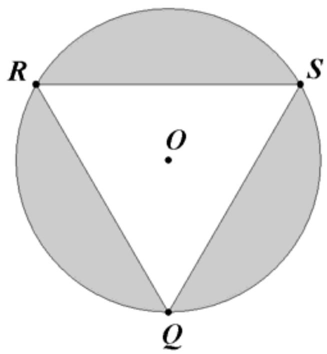

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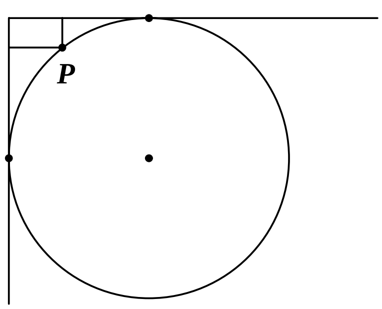
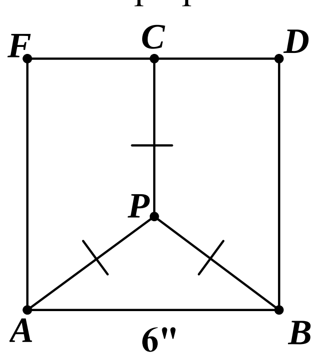
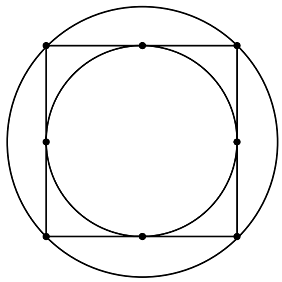


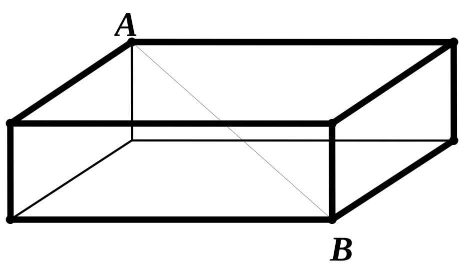
10. A square with 6" sides is shown. If P is a point in the interior of the square such that the segments $\overline{PA}$, $\overline{PB}$, and $\overline{PC}$ are equal in length, and $\overline{PC}$ is perpendicular to $\overline{FD}$, what is the area, in square inches, of ΔAPB?



11. The square shown is externally tangent to the smaller circle and internally tangent to the larger circle. Find the ratio of the area of the smaller circle to the area of the larger circle.

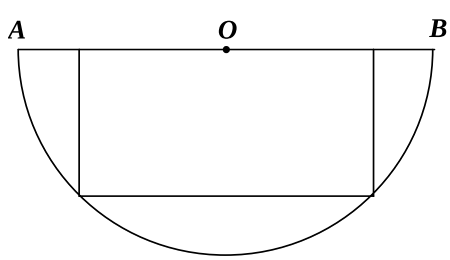


12. A rectangular solid has dimensions 3 cm by 4 cm by 12 cm. What is the length, in centimeters, of the diagonal $\overline{AB}$ of the solid?



13. Point A is located in 3-dimensional space at coordinate position (4, 4, 4). Find the distance from point A to the origin.

14. The semicircle shown has center O and diameter $\overline{AB}$. A rectangle whose length is twice its width has two vertices on the semicircle. Find the ratio of the area of the semicircle to the area of the rectangle.



References and Additional Resources

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