

Unit 3 – Quadrilaterals

Isosceles Right Triangle Reflections

Overview: Participants develop the properties of squares through reflections of isosceles right triangles.

Objective: **TEExES Mathematics Competencies**
III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
III.013.A. The beginning teacher analyzes the properties of polygons and their components.
III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).
III.014.B. The beginning teacher uses the properties of transformations and their compositions to solve problems.
III.014.D. The beginning teacher applies transformations in the coordinate plane.
V.018.B. The beginning teacher applies correct mathematical reasoning to derive valid conclusions from a set of premises.
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.
V.019.F. The beginning teacher uses appropriate mathematical terminology to express mathematical ideas.

Geometry TEKS

b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
d.2.A. The student uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures.
e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of polygons and their component parts.
e.3.A. The student uses congruence transformations to make conjectures and justify properties of geometric figures.

f.1. The student uses similarity properties and transformations to explore and justify conjectures about geometric figures.

Background: Participants need to have a knowledge of transformations.

Materials: colored pencils, easel paper, colored markers, centimeter ruler, transparency

New Terms: rotational symmetry

Procedures:

In the Quadrilaterals unit, the properties of triangles and transformations are used to develop the properties of quadrilaterals.

Participants work the activity individually but verify solutions informally with group members. Facilitate by providing minimal prompts to help participants clarify their thinking. Try *not* to answer questions directly.

Bertrand Russell said, “...what matters in mathematics...is not the intrinsic nature of our terms but the logical nature of their interrelations.” The quadrilateral unit explores the interrelations of triangles and quadrilaterals, using transformations as a tool to construct quadrilaterals.

Mark isosceles right $\triangle ABC$ with the known properties in terms of angles and sides, using tick marks and colors.

Verify the properties of angles and sides with your group members.

The isosceles right triangle has a right angle, $\angle B$, two congruent sides, \overline{AB} and \overline{BC} , $\overline{AB} \perp \overline{BC}$, and two congruent angles $\angle A$ and $\angle C$, each 45° .

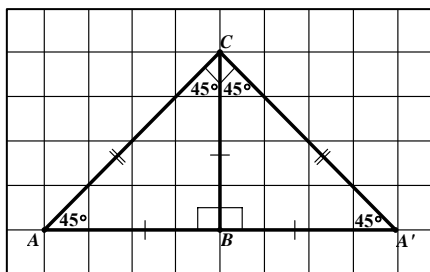
Reflect $\triangle ABC$ across the line containing \overline{BC} . Label image vertices and properties appropriately.

1. What type of triangle is $\triangle ACA'$? Justify your answer.

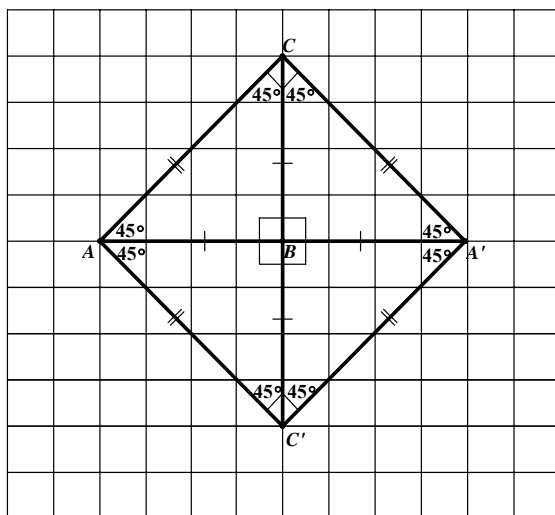
$\triangle ACA'$ is an isosceles right triangle.

$$AC = CA'$$

$$m\angle ACA' = m\angle ACB + m\angle A'CB = 90^\circ.$$



2. Reflect $\triangle ACA'$ and its component parts across the line containing $\overline{AA'}$. All of the properties of $\triangle ACA'$ apply to the reflected triangle. Label the properties of quadrilateral $ACA'C'$.



3. Classify the quadrilateral $ACA'C'$, formed from the composite reflections of an isosceles right triangle.
 $ACA'C'$ is a square. The figure has four right angles and four congruent sides.
4. List the properties of quadrilateral $ACA'C'$ in terms of the sides, angles, diagonals and symmetry in the table below.

Possible properties follow.

Sides:

- *All four sides are congruent.*
- *Opposite sides are parallel because alternate interior angles are congruent.*
Using symbols, $\overline{AC} \parallel \overline{A'C'}$ and $\overline{CA'} \parallel \overline{AC'}$.
- *Consecutive sides are perpendicular, because vertex angles of the figure are all right angles, for example, $\overline{AC'} \perp \overline{A'C'}$.*

Vertex angles:

- *All four angles are congruent right angles.*
- *The vertex angles are bisected by the diagonals.*
- *Opposite angles are congruent and supplementary.*
- *Consecutive angles are congruent and supplementary.*

Diagonals:

- *There are two diagonals.*
- *Diagonals are congruent to each other.*
- *Diagonals bisect each other at right angles.*
- *Diagonals bisect the vertex angles of the square.*
- *Diagonals lie on two of the lines of symmetry for the figure.*

Symmetry:

- *There are four lines of symmetry.*
- *The diagonals lie on two of the lines of symmetry.*
- *The other two lines of symmetry pass through the midpoints of the sides of the square.*
- *90° (4-fold) rotational symmetry exists. The figure can be rotated 90° so that the resulting image coincides with the original image. When the figure is rotated four times through 90° , the original vertices coincide.*

In general, a figure has *rotational symmetry* if there is a rotation that results in the image superimposing on the pre-image. Remind participants to add the new term rotational symmetry to their glossaries.

To complete the activity, each group draws the figure and lists its properties on a sheet of easel paper. The sheet of easel paper is placed on the wall for a gallery walk. Pairs of groups view each other's work. Allow groups about 5 minutes to meet and discuss any differences or errors on the posters.

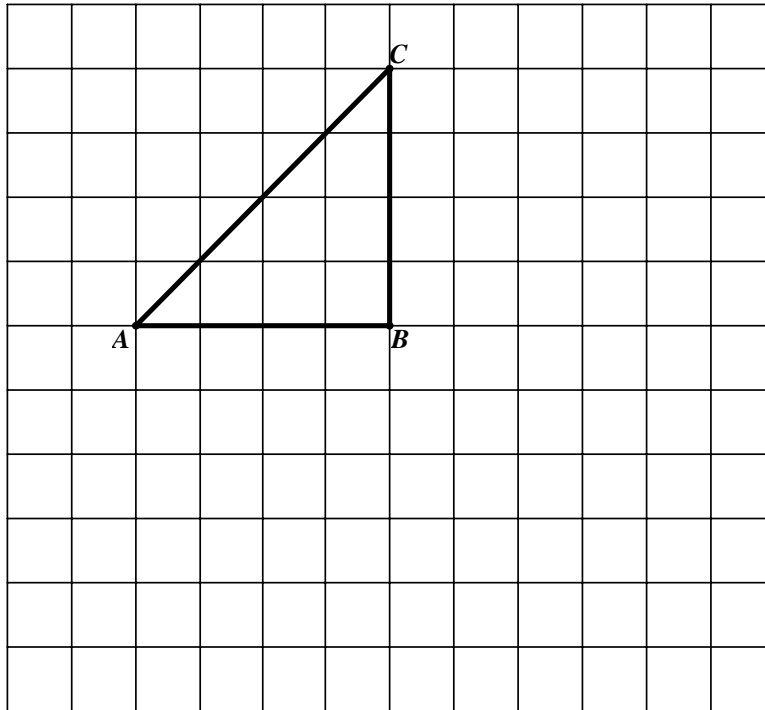
Bring participants together for a whole class discussion. Summarize the properties of squares. Leave the posters on the wall. At the conclusion of the quadrilateral unit, participants can compare the properties of the square, rhombus, kite, rectangle, parallelogram and trapezoid.

Participants are performing at the van Hiele Descriptive Level because they develop properties of squares.

Isosceles Right Triangle Reflections

Bertrand Russell said, "...what matters in mathematics...is not the intrinsic nature of our terms but the logical nature of their interrelations." The quadrilateral unit explores the interrelations of triangles and quadrilaterals, using transformations as a tool to construct quadrilaterals.

Mark isosceles right $\triangle ABC$ with the known properties in terms of angles and sides, using tick marks and colors.



Verify the properties of angles and sides with your group members.

Reflect $\triangle ABC$ across the line containing \overline{BC} , and label image vertices and properties appropriately.

1. What type of triangle is $\triangle ACA'$? Justify your answer.
2. Reflect $\triangle ACA'$ and its component parts across the line containing $\overline{AA'}$. All of the properties of $\triangle ACA'$ apply to the reflected triangle. Label the properties of quadrilateral $ACA'C'$.

- Classify the quadrilateral $ACA'C'$, formed from the composite reflections of an isosceles right triangle.
- List the properties of quadrilateral $ACA'C'$ in terms of the sides, angles, diagonals and symmetry in the table below.

Sides	Vertex Angles
Diagonals	Symmetry

Scalene Right Triangle Reflections

Overview: Participants develop the properties of rhombi through reflections of scalene right triangles.

Objective: **TE_xES Mathematics Competencies**

III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.

III.013.A. The beginning teacher analyzes the properties of polygons and their components.

III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).

III.014.B. The beginning teacher uses the properties of transformations and their compositions to solve problems.

III.014.D. The beginning teacher applies transformations in the coordinate plane.

V.018.B. The beginning teacher applies correct mathematical reasoning to derive valid conclusions from a set of premises.

V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.

V.019.F. The beginning teacher uses appropriate mathematical terminology to express mathematical ideas.

Geometry TEKS

b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

b.3.B. The student constructs and justifies statements about geometric figures and their properties.

b.3.D. The student uses inductive reasoning to formulate a conjecture.

c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.

d.2.A. The student uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures.

e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties of parallel and perpendicular lines.

e.3.A. The student uses congruence transformations to make conjectures and justify properties of geometric figures.

f.1. The student uses similarity properties and transformations to explore and justify conjectures about geometric figures.

Background: Participants need to have a knowledge of transformations.

Materials: colored pencils, easel paper, graph paper, colored markers, centimeter ruler

New Terms:

Procedures:

Distribute the activity sheet. Ask participants to complete the activity individually but informally verify with group members. While participants work, walk around the room listening and facilitating. Try *not* to directly answer questions, but rather provide minimal prompts to help participants clarify their thinking.

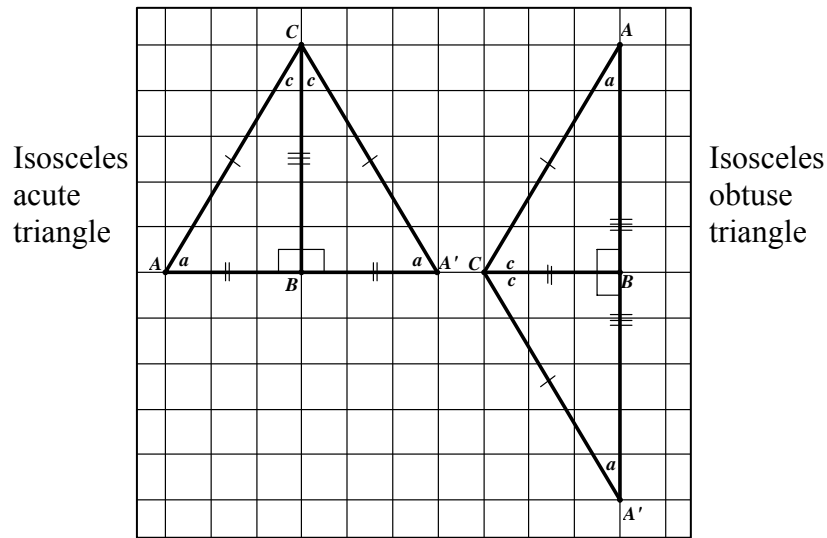
When the group members complete the activity, each group draws the figure and lists its properties on a sheet of easel paper. The sheet of easel paper is placed on the wall for a gallery walk. Allow pairs of groups about 5 minutes to meet and discuss any differences or errors on the posters.

In this activity, a scalene right triangle is reflected over the line containing one of the legs of the triangle, then the composite figure is reflected over the line containing the other leg. Predict which quadrilateral will be created.

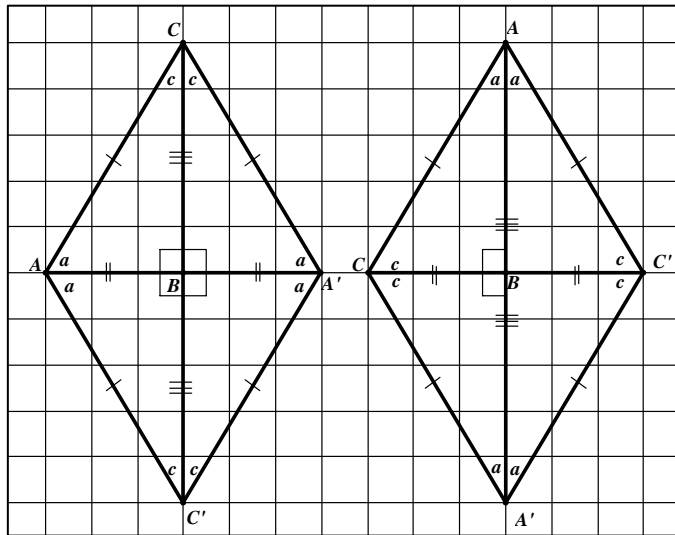
In the middle of a sheet of graph paper, draw a scalene right triangle, $\triangle ABC$, with the right angle at B . Make sure that the legs of the triangle lie along grid lines, and that the vertices are located on grid line intersections. The legs of the triangle should be 2 to 3 inches long. Mark the sides and angles with known properties of scalene right triangles. Use different tick marks to indicate non-congruency.

Reflect $\triangle ABC$ across the line containing \overline{BC} . Label image vertices with prime marks.

1. What type of triangle is $\triangle ACA'$? Why? Discuss with your group to make sure there is agreement.
 $\triangle ACA'$ is an isosceles acute triangle or an isosceles obtuse triangle. The congruent legs and congruent base angles are formed as a result of the reflection. The base angles are not 45° angles, and the vertex angle at C is not a right angle.



2. Reflect $\triangle ACA'$ and its component parts across the line containing $\overline{AA'}$. Label the properties of quadrilateral $ACA'C'$. Discuss with your group to make sure there is agreement.
3. Classify the quadrilateral $ACA'C'$. Justify.
Quadrilateral $ACA'C'$ is a rhombus because it has four congruent sides.



4. List the properties of quadrilateral $ACA'C'$ in terms of the sides, angles, diagonals and symmetry in the table below.

Sides:

- *All four sides are congruent.*
- *Opposite sides are parallel because alternate interior angles are congruent.*

Vertex Angles:

- *Opposite angles are congruent.*
- *Consecutive angles are supplementary because the pre-image acute angles are complementary. The consecutive angles, which are composed of two sets of the acute complementary angles, must be supplementary.*

Diagonals:

- *There are two diagonals.*
- *Diagonals are not congruent to each other.*
- *Diagonals bisect each other.*
- *Diagonals intersect at 90° angles.*
- *Diagonals bisect the vertex angles of the rhombus, because the vertex angles were formed by reflection.*
- *Diagonals are lines of symmetry of the figure, because they lie on the original reflection lines.*

Symmetry:

- *There are two lines of symmetry.*
- *The diagonals lie on the two lines of symmetry, passing through opposite vertices of the rhombus.*

Bring participants together for a whole class discussion. Compare the properties of the rhombus with the properties of the square. Possible comparisons follow:

- Both figures have four congruent sides.
- In both quadrilaterals opposite vertex angles are congruent and consecutive vertex angles are supplementary.
- In both quadrilaterals the diagonals bisect each other at right angles, and bisect the vertex angles.
- In both quadrilaterals the diagonals lie on lines of symmetry.

As the properties of each quadrilateral are listed, the posters remain on the wall so that the properties of squares, kites, rectangles, parallelograms and trapezoids can be compared and contrasted.

Participants are performing at the van Hiele Descriptive Level because properties of a rhombus are being developed. In the discussion comparing the properties of the square and rhombus participants approach the Relational Level.

Scalene Right Triangle Reflections

In this activity, a scalene right triangle is reflected over the line containing one of the legs of the triangle, then the composite figure is reflected over the line containing the other leg. Predict which quadrilateral will be created.

In the middle of a sheet of graph paper, draw a scalene right triangle, $\triangle ABC$, with the right angle at B . Make sure that the legs of the triangle lie along grid lines, and that the vertices are located on grid line intersections. The legs of the triangle should be 2 to 3 inches long. Mark the sides and angles with known properties of scalene right triangles. Use different tick marks to indicate non-congruency.

Reflect $\triangle ABC$ across the line containing \overline{BC} . Label image vertices with prime marks.

1. What type of triangle is $\triangle ACA'$? Why? Discuss with your group to make sure there is agreement.

2. Reflect $\triangle ACA'$ and its component parts across the line containing $\overline{AA'}$. Label the properties of quadrilateral $ACA'C'$. Discuss with your group to make sure there is agreement.

3. Classify the quadrilateral $ACA'C'$. Justify.

4. List the properties of quadrilateral $ACA'C'$ in terms of the sides, angles, diagonals and symmetry in the table below.

Sides	Vertex Angles
Diagonals	Symmetry

Scalene Acute/Obtuse Triangle Reflections

Overview: Participants develop the properties of kites through reflections of scalene acute or scalene obtuse triangles.

Objective: **TE_xES Mathematics Competencies**
III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
III.013.A. The beginning teacher analyzes the properties of polygons and their components.
III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).
III.014.D. The beginning teacher applies transformations in the coordinate plane.
V.018.B. The beginning teacher applies correct mathematical reasoning to derive valid conclusions from a set of premises.
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.
V.019.F. The beginning teacher uses appropriate mathematical terminology to express mathematical ideas.

Geometry TEKS

b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
d.2.A. The student uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures.
e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of polygons and their component parts.
e.3.A. The student uses congruence transformations to make conjectures and justify properties of geometric figures.
f.1. The student uses similarity properties and transformations to explore and justify conjectures about geometric figures.

Background: Participants need to have a knowledge of transformations for this activity.

Materials: colored pencils, easel paper, colored markers, centimeter ruler

New Terms:

Procedures:

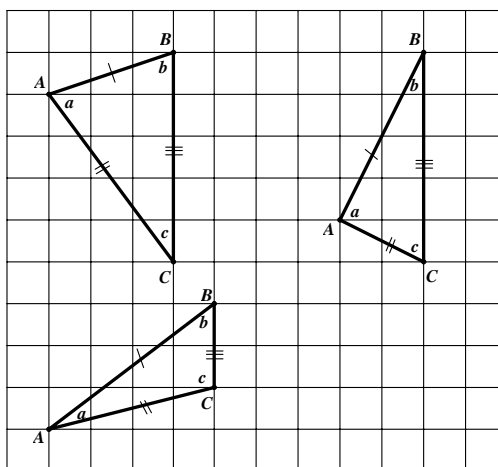
Distribute the activity sheet. Participants work the activity individually, but informally verify with group members. Facilitate by providing minimal prompts to help participants clarify their thinking. Try *not* to answer questions directly.

Each group draws the figure and lists its properties on a sheet of easel paper. When completed the sheet of easel paper is posted on the wall for a gallery walk. Give pairs of groups about 5 minutes to meet and discuss any differences or errors on the posters.

In this activity, a scalene acute triangle or a scalene obtuse triangle is reflected across the line containing one of its sides. Predict what quadrilateral will be created.

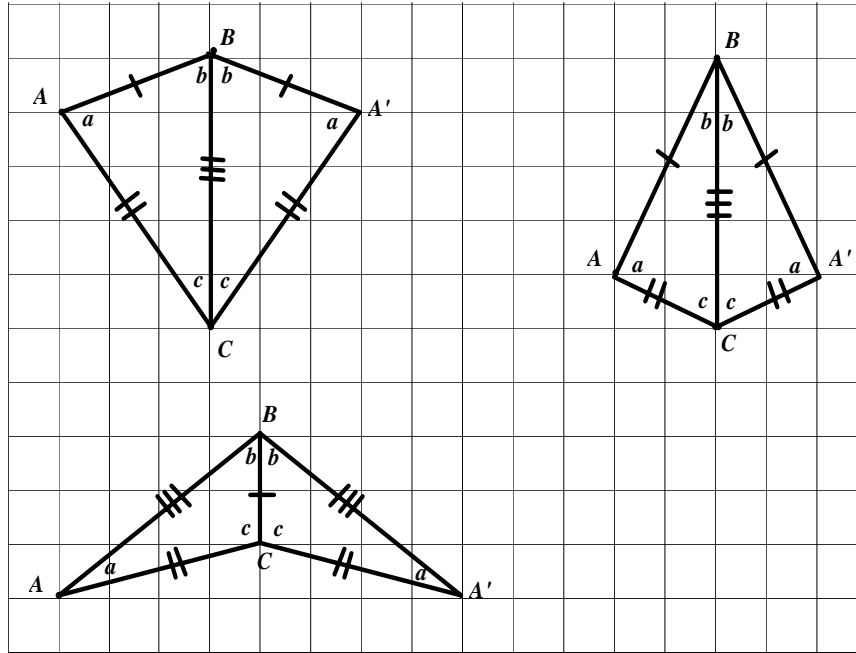
In your group decide who will draw a scalene acute triangle and who will draw a scalene obtuse triangle. In the middle of a sheet of graph paper, draw $\triangle ABC$ so that \overline{BC} coincides with a grid line and all three vertices lie at grid line intersections. The sides of the triangle should be 2 to 3 inches long. Mark the sides and angles using different tick marks to indicate non-congruency.

Possible examples are shown.



Reflect $\triangle ABC$ across the line containing \overline{BC} , and label the image appropriately.

1. Classify quadrilateral $ACA'B$. Justify.
ACA'B is a kite. A kite is a quadrilateral with exactly two distinct pairs of congruent consecutive sides. The term diamond is sometimes used at the Visual Level.



2. List the properties of quadrilateral $ACA'B$ in terms of the sides, angles, diagonals and symmetry in the table below.

Sides:

- *Two pairs of sides are congruent.*
- *Two sets of consecutive sides are congruent.*
- *Opposite sides are not congruent.*

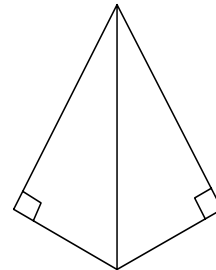
Note: Opposite sides are not parallel because alternate interior angles are not congruent.

Vertex angles:

- *Only one pair of opposite angles is congruent. One pair of opposite angles is not congruent.*
- *Consecutive angles are not congruent.*
- *The non-congruent vertex angles are bisected by one of the diagonals. The congruent vertex angles are not bisected by a diagonal.*

Is it possible for a kite to have a pair of right angles?

The figures for this activity were constructed from acute or obtuse triangles, but if a scalene right triangle is reflected across the line containing its hypotenuse, then the congruent pair of vertex angles are right angles.



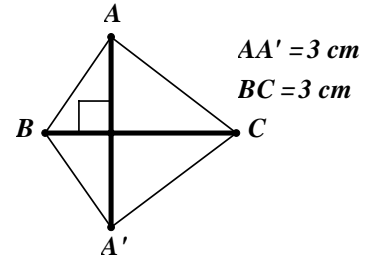
Diagonals:

- *There are two diagonals.*
- *One diagonal lies on the line of symmetry.*

- The diagonal lying on the line of symmetry bisects the other diagonal at right angles.
- Diagonals may not be congruent to each other.

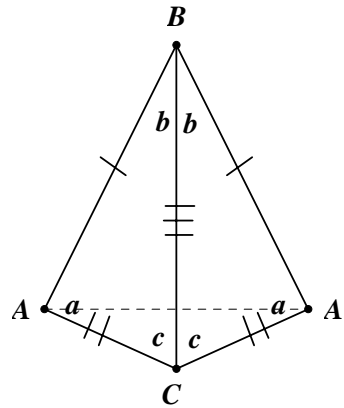
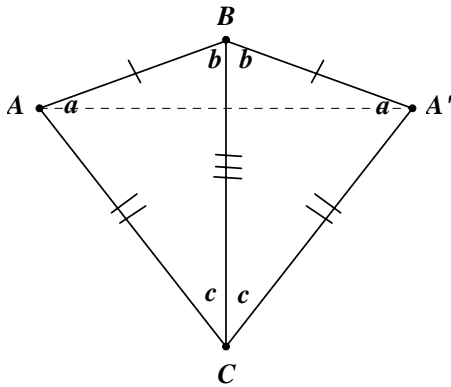
Is it possible for the kite to have congruent diagonals?

Yes. The figure to the right is an example of a kite with congruent diagonals.

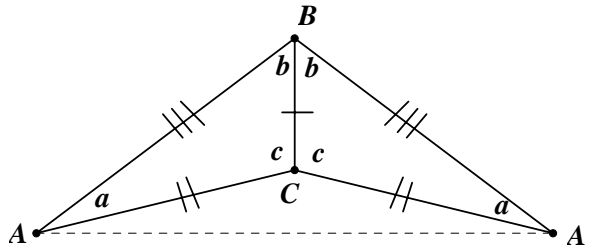


Note: Ask participants to look at the triangles formed on either side of $\overline{AA'}$, the diagonal which does not lie on the line of symmetry. These triangles, $\triangle ABA'$ and $\triangle ACA'$, are both isosceles triangles.

The kites below are both convex.



The kite below is concave.



Symmetry:

- There is one line of symmetry.
- The line of symmetry contains one of the diagonals.

Bring participants together for a whole class discussion. Summarize the properties of kites.

Participants are performing at the van Hiele Descriptive Level because they develop properties of a kite.

Scalene Acute/Obtuse Triangle Reflections

In this activity a scalene acute triangle or a scalene obtuse triangle is reflected across the line containing one of its sides. Predict what quadrilateral will be created.

In your group decide who will draw a scalene acute triangle and who will draw a scalene obtuse triangle. In the middle of a sheet of graph paper, draw $\triangle ABC$ so that \overline{BC} coincides with a grid line and all three vertices lie at grid line intersections. The sides of the triangle should be 2 to 3 inches long. Mark the sides and angles using different tick marks to indicate non-congruency.

Reflect $\triangle ABC$ across the line containing \overline{BC} , and label the image appropriately.

1. Classify quadrilateral $ACA'B$. Justify.

2. List the properties of quadrilateral $ACA'B$ in terms of the sides, angles, diagonals and symmetry in the table below.

Sides	Vertex Angles
Diagonals	Symmetry

Rotate a Triangle

Overview: Participants discover properties of rectangles by rotating a right triangle around the midpoint of its hypotenuse, and discover the properties of parallelograms by rotating a non-right triangle around the midpoint of one of its sides.

Objective: **TEXES Mathematics Competencies**

III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.

III.013.A. The beginning teacher analyzes the properties of polygons and their components.

III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).

III.014.D. The beginning teacher applies transformations in the coordinate plane.

V.018.B. The beginning teacher applies correct mathematical reasoning to derive valid conclusions from a set of premises.

V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.

V.019.F. The beginning teacher uses appropriate mathematical terminology to express mathematical ideas.

Geometry TEKS

b.2.B. Makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

b.3.B. The student constructs and justifies statements about geometric figures and their properties.

c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.

d.2.A. The student uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures.

e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of circles and the lines that intersect them.

e.3.A. The student uses congruence transformations to make conjectures and justify properties of geometric figures.

f.1. The student uses similarity properties and transformations to explore and justify conjectures about geometric figures.

Background: Participants need a knowledge of the properties of rotation and parallel lines.

Materials: easel paper, graph paper, colored markers, patty paper, centimeter ruler

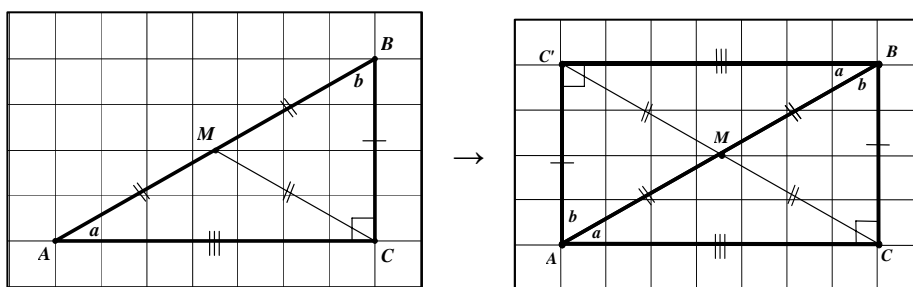
New Terms:

Procedures:

Distribute the activity sheets. Participants work on all seven items in groups. During the whole class discussion, ask two volunteers to record properties of rectangles and parallelograms on separate sheets of easel paper.

1. In the middle of a sheet of graph paper draw scalene right $\triangle ABC$, with the right angle at vertex C . Draw the legs along grid lines and locate the vertices at grid line intersections. The lengths of the legs of the triangles should be 2 to 3 inches long. Locate the midpoint, M , of the hypotenuse. Draw the median to the hypotenuse. Label the figure appropriately to indicate congruence or non-congruence.
2. Rotate $\triangle ABC$ 180° around M . Label the image appropriately.

The figures below represent the process.



3. In your group discuss and list the properties of rectangle $ACBC'$.

The following is a list of possible properties for the rectangle:

Sides:

- *Opposite sides are congruent. (Rotation preserves congruence.)*
- *Opposite sides are parallel. (The consecutive angles are supplementary. If the interior angles on the same side of a transversal are congruent, then the lines intersected by the transversal are parallel.)*
- *Consecutive sides are perpendicular to each other.*

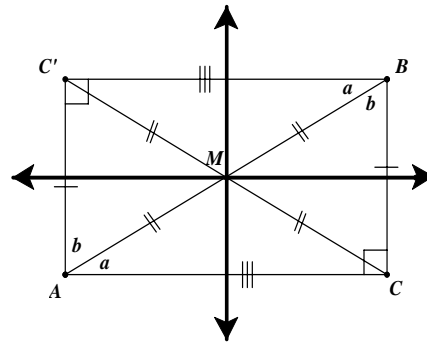
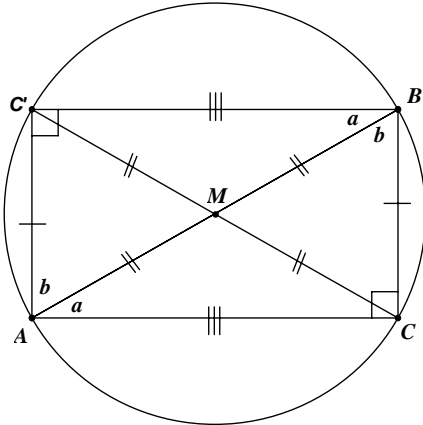
Vertex angles:

- *All vertex angles are congruent right angles.*
- *Opposite angles are congruent and supplementary.*

- *Consecutive angles are congruent and supplementary.*

Diagonals:

- *Diagonals bisect each other.*
- *Diagonals are congruent.*
- *The point of intersection of the diagonals is also the center of the circumscribed circle.*



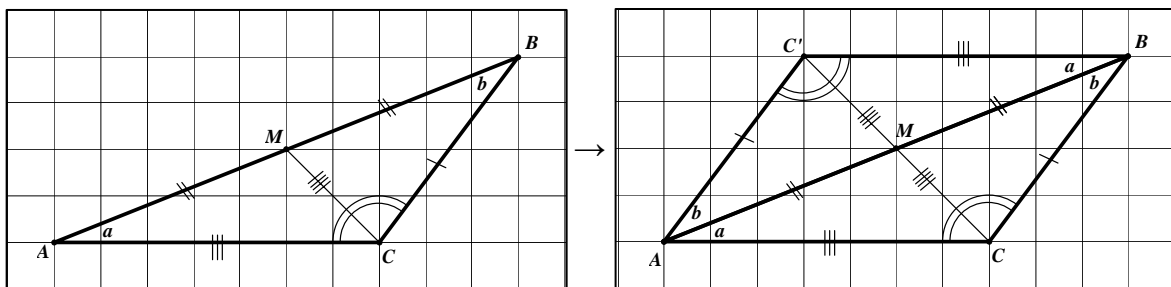
Symmetry:

- *The rectangle has two symmetry lines.*
- *The rectangle has 180° (or 2-fold) rotational symmetry.*

Participants may argue that the diagonals are symmetry lines. To clear up this misconception, trace the figure on patty paper and fold along the diagonals.

- On a clean sheet of graph paper draw obtuse or acute scalene $\triangle ABC$, with one of its sides along one of the grid lines. Locate the vertices at grid line intersections. The sides of the triangle should be 1.5 to 3 inches long. Locate the midpoint, M , of \overline{AB} . Draw the median to side \overline{AB} . Label the figure appropriately indicating congruence or non-congruence.
- Rotate $\triangle ABC$ 180° around M . Label the image appropriately.

The figures below represent examples.



6. In your group discuss and list the properties of parallelogram $ACBC'$ under the given headings.

Sides:

- *Opposite sides are congruent. (Rotation preserves congruence.)*
- *Opposite sides are parallel. (Alternate interior angles are congruent, because rotation preserves congruence.)*

Vertex angles:

- *Opposite vertex angles are congruent. (Rotation preserves congruence.)*
- *Consecutive angles are supplementary. (Interior angles on the same side of a transversal that intersects parallel lines are supplementary.)*

Diagonals:

- *Diagonals bisect each other. (M is the midpoint of \overline{AB} and \overline{MC} is mapped to $\overline{MC'}$, so that M the midpoint of CC' .)*

Symmetry:

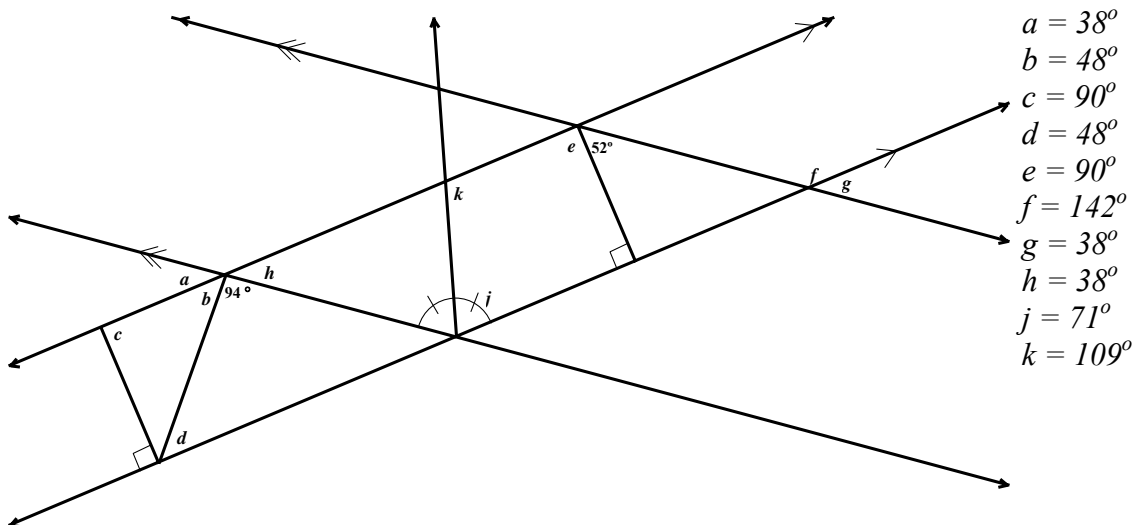
- *The parallelogram has 180° (or 2-fold) rotation. (The figure was produced using 180° rotation.)*

Participants may argue that the diagonals are symmetry lines. To clear up this misconception, trace the figure on patty paper and fold along the diagonals.

7. Compare the properties of the parallelogram with the properties of the rectangle.
In both quadrilaterals opposite sides are parallel and congruent; opposite angles are congruent; diagonals bisect each other.

Application problems:

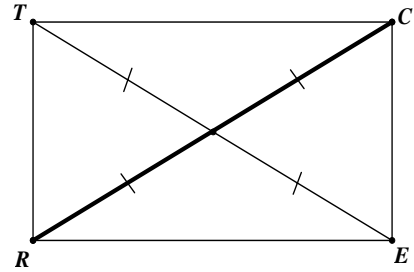
8. Calculate the measure of each lettered angle.



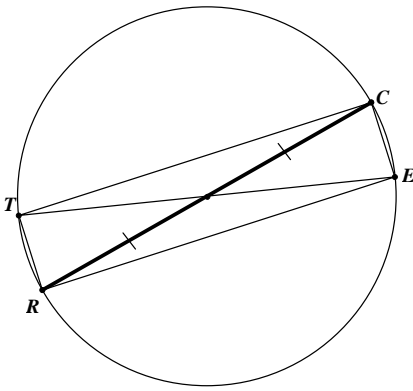
9. \overline{RC} is a diagonal of rectangle $RECT$. Where can the other two vertices, E and T , be located?

The diagonals are congruent and intersect each other at their respective midpoints. The other diagonal can be any congruent line segment, whose midpoint is also the midpoint of the given segment.

Find the midpoint of \overline{RC} . Draw a segment from the midpoint to a point not on \overline{RC} , congruent to one half of \overline{RC} . Extend the segment an equal distance on the opposite side of the midpoint.



Alternatively: Draw a circle using \overline{RC} as the diameter. \overline{ET} is also a diameter.



Participants are performing at the van Hiele Descriptive Level because properties of rectangles and parallelograms are being developed. The comparison of the properties of the rectangle and parallelogram approaches the Relational Level.

In 7, participants apply the properties of parallelograms and rectangles at the Descriptive Level.

In 8, the first solution requires the Descriptive Level. In the second solution, participants combine two figures with related properties, thus approaching the Relational Level.

Rotate a Triangle

1. In the middle of a sheet of graph paper draw scalene right $\triangle ABC$, with the right angle at vertex C . Draw the legs along grid lines and locate the vertices at grid line intersections. The lengths of the legs of the triangle should be 2 to 3 inches long. Locate the midpoint, M , of the hypotenuse. Draw the median to the hypotenuse. Label the figure appropriately to indicate congruence or non-congruence.
2. Rotate $\triangle ABC$ 180° around M . Label the image appropriately.
3. In your group discuss and list the properties of rectangle $ACBC'$.

Sides	Vertex Angles
Diagonals	Symmetry

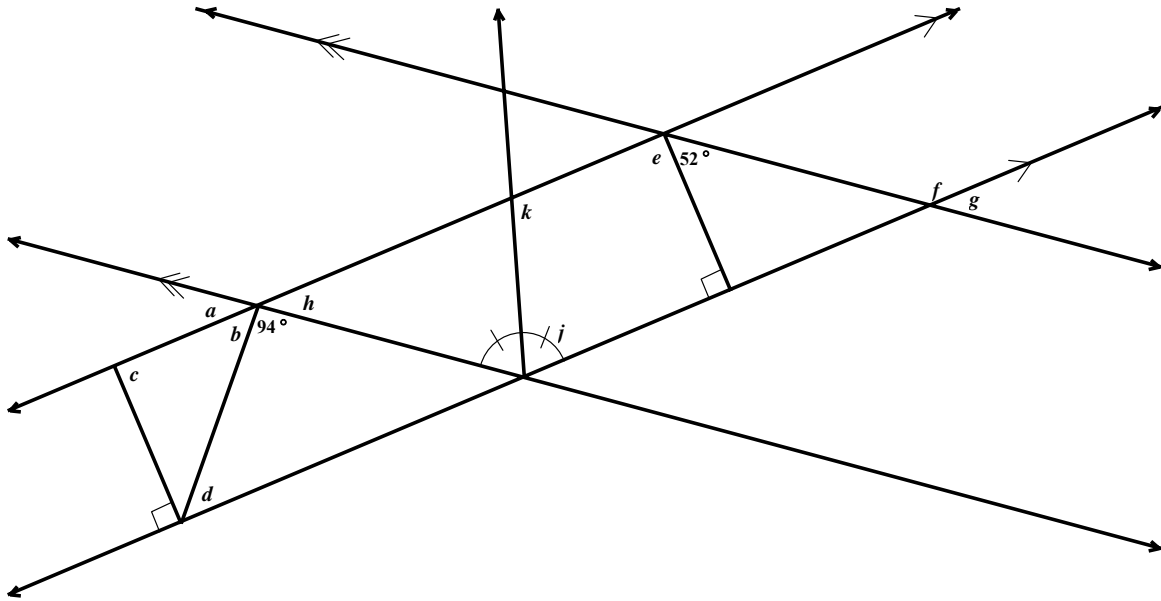
- On a clean sheet of graph paper draw obtuse or acute scalene $\triangle ABC$, with one of its sides along one of the grid lines. Locate the vertices at grid line intersections. The sides of the triangle should be 1.5 to 3 inches long. Locate the midpoint, M , of \overline{AB} . Draw the median to side \overline{AB} . Label the figure appropriately indicating congruence or non-congruence.
- Rotate $\triangle ABC$ 180° around M . Label the image appropriately.
- In your group discuss and list the properties of parallelogram $ACBC'$ under the given headings.

Sides	Vertex Angles
Diagonals	Symmetry

- Compare the properties of the parallelogram with the properties of the rectangle.

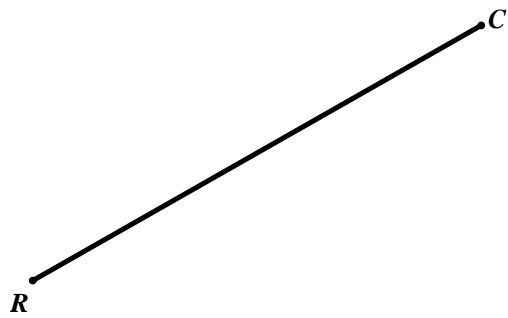
Application problems:

8. Calculate the measure of each lettered angle.



$a = \underline{\hspace{1cm}}$ $b = \underline{\hspace{1cm}}$ $c = \underline{\hspace{1cm}}$ $d = \underline{\hspace{1cm}}$ $e = \underline{\hspace{1cm}}$
 $f = \underline{\hspace{1cm}}$ $g = \underline{\hspace{1cm}}$ $h = \underline{\hspace{1cm}}$ $j = \underline{\hspace{1cm}}$ $k = \underline{\hspace{1cm}}$

9. \overline{RC} is a diagonal of rectangle $RECT$. Where can the other two vertices, E and T , be located?



Truncate a Triangle's Vertex

Overview: Participants discover the properties of trapezoids.

Objective: **TEXES Mathematics Competencies**

II.006.B. The beginning teacher writes equations of lines given various characteristics (e.g., two points, a point and slope, slope and y -intercept).

II.006.G. The beginning teacher models and solves problems involving linear and quadratic equations and inequalities using a variety of methods, including technology.

III.011.A. The beginning teacher applies dimensional analysis to derive units and formulas in a variety of situations (e.g., rates of change of one variable with respect to another) and to find and evaluate solutions to problems.

III.011.B. The beginning teacher applies formulas for perimeter, area, surface area, and volume of geometric figures and shapes (e.g., polygons, pyramids, prisms, cylinders, cones, spheres) to solve problems.

III.012.B. The beginning teacher uses properties of points, lines, planes, angles, lengths, and distances to solve problems.

III.012.C. The beginning teacher applies the properties of parallel and perpendicular lines to solve problems.

III.012.E. The beginning teacher describes and justifies geometric constructions made using compass and straightedge, reflection devices, and other appropriate technologies.

III.013.A. The beginning teacher analyzes the properties of polygons and their components.

III.014.E. The beginning teacher applies concepts and properties of slope, midpoint, parallelism, perpendicularity, and distance to explore properties of geometric figures and solve problems in the coordinate plane.

V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.

V.018.F. The beginning teacher evaluates how well a mathematical model represents a real-world situation.

V.019.B. The beginning teacher understands how mathematics is used to model and solve problems in other disciplines (e.g., art, music, science, social science, business).

V.019.D. The beginning teacher communicates mathematical ideas using a variety of representations (e.g., numeric, verbal, graphical, pictorial, symbolic, concrete).

V.019.F. The beginning teacher uses appropriate mathematical terminology to express mathematical ideas.

Geometry TEKS

- b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.
- b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
- b.3.B. The student constructs and justifies statements about geometric figures and their properties.
- b.3.D. The student uses inductive reasoning to formulate a conjecture.
- c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
- d.2.A. The student uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures.
- d.2.B. The student uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons.
- d.2.C. The student develops and uses formulas including distance and midpoint.
- e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of polygons and their component parts.

Background: Participants need a knowledge of properties of isosceles triangles, parallel lines, and their associated angles.

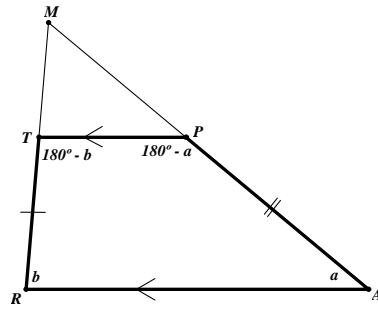
Materials: easel paper, graph paper, colored markers, centimeter ruler

New Terms: isosceles trapezoid, midsegment

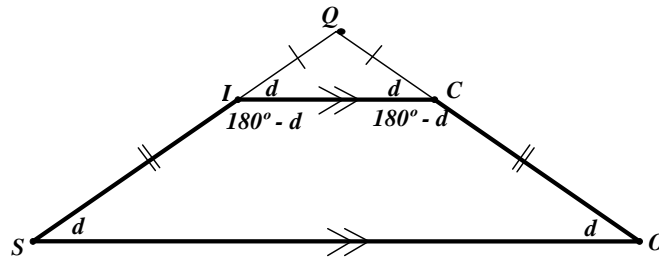
Procedures:

Distribute the activity sheet. Participants complete 1 – 8. They may work independently or in groups. 1 and 2 provide information and vocabulary needed to define a trapezoid and an isosceles trapezoid.

1. Draw a large scalene triangle. Label it $\triangle MRA$. Locate a point T on \overline{MR} . Construct a line parallel to \overline{RA} , through point T , which intersects \overline{MA} at point P . $TRAP$ is a trapezoid. The parallel sides \overline{RA} and \overline{TP} are called the bases of the trapezoid. In your figure clearly identify the angle relations within the trapezoid.



2. Draw a large isosceles triangle. Label it $\triangle QSO$ with vertex angle $\angle Q$. Locate a point I on \overline{SQ} . Construct a line parallel to side \overline{SO} through point I , which intersects \overline{QO} at point C . $ISOC$ is an isosceles trapezoid. The congruent pairs of angles in an isosceles trapezoid are called base angles. In your figure, clearly identify the angle relations within the trapezoid.



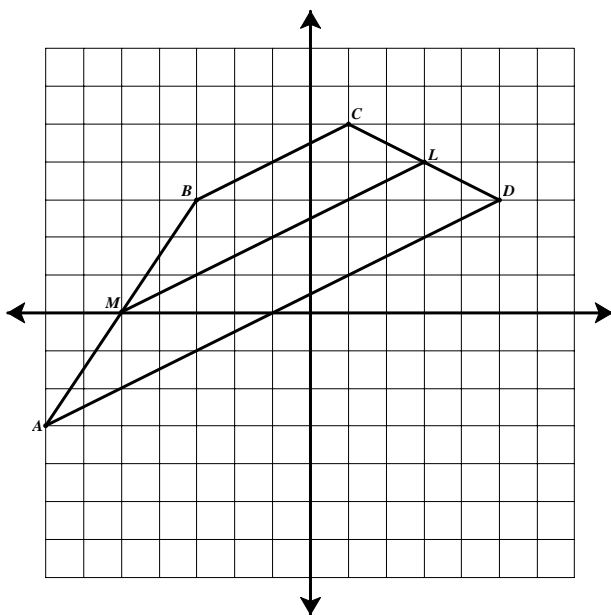
An *isosceles trapezoid* is a trapezoid with congruent legs.

3. Work with your group to find the angle and side properties of trapezoids and isosceles trapezoids, using your knowledge of parallel lines, isosceles and scalene triangles. Be prepared to share and justify the properties during whole class discussion. Note that some books define a trapezoid as a quadrilateral with “at least one pair of parallel sides.” Others define a trapezoid as a quadrilateral with “only one pair of parallel sides”. This module will use the latter definition.

Possible properties:

- A trapezoid has one pair of parallel sides.
- The sum of the angles of a trapezoid is 360° .
- The pairs of consecutive angles at opposite bases are supplementary. (Interior angles on the same side of a transversal that intersects parallel lines are supplementary.)
- In an isosceles trapezoid the base angles (two pairs) are congruent.
- In an isosceles trapezoid the non-parallel sides are congruent.

4. Locate the vertices of quadrilateral $ABCD$ on the coordinate plane at $A(-7, -3)$, $B(-3, 3)$, $C(1, 5)$, and $D(5, 3)$. Explain why $ABCD$ is a trapezoid.



Sides \overline{BC} and \overline{AD} are parallel because their slopes are both $\frac{1}{2}$. The other two sides have unequal slopes.

5. Find the midpoints, M and L , of \overline{AB} and \overline{CD} respectively. \overline{ML} is called a midsegment. $M(-5, 0)$, $L(3, 4)$.

A *midsegment* is a line segment with endpoints that are the midpoints of the legs of the trapezoid.

6. How do the coordinates of a midpoint of a segment relate to the coordinates of its endpoints?
The coordinates of the midpoint are the averages of the coordinates of the endpoints:
- $$M(-5, 0) = \left(\frac{-7 - 3}{2}, \frac{-3 + 3}{2} \right); L(3, 4) = \left(\frac{1 + 5}{2}, \frac{5 + 3}{2} \right).$$

7. How does the slope of the midsegment relate to the slopes of the parallel sides of the trapezoid?

The slope of the midsegment is $\frac{1}{2}$, which is the same as the slopes of the bases.

Therefore the midsegment is parallel to the bases.

8. How does the length of the midsegment relate to the lengths of the parallel sides of the trapezoid?

The length of midsegment \overline{ML} is the average of the lengths of \overline{BC} and \overline{AD} .

When most groups have completed 1 – 8, conduct a whole class discussion on the properties of trapezoids. Participants justify each property. Possible properties, with justifications in parentheses follow:

- *A trapezoid has exactly one pair of parallel sides (shown by the construction).*

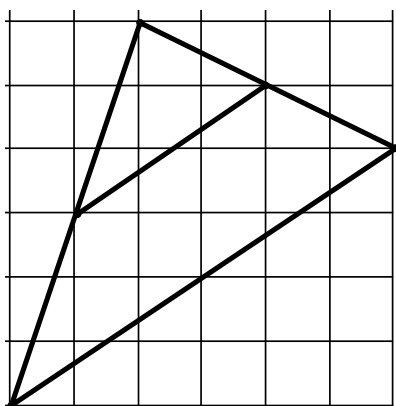
- *The sum of the angles is 360° . (There are two sets of interior supplementary angles between parallel lines.)*
- *The pairs of consecutive angles at opposite bases are supplementary. (The interior angles on the same side of a transversal that intersects parallel lines are supplementary.)*
- *In an isosceles trapezoid, the two pairs of base angles are congruent. (The two base angles from the original isosceles triangle are congruent; the other two angles are supplementary to the original two base angles, and must also be congruent to each other.)*
- *In an isosceles trapezoid, the non-parallel sides are congruent. (Using 2 as an example, the base angles are congruent and congruent to the corresponding base angles of $\triangle QIC$. Therefore, $\triangle QIC$ is isosceles and $\overline{QI} \cong \overline{QC}$. Since $\overline{QS} \cong \overline{QO}$, the congruent sides of $\triangle QSO$, then by subtraction, $\overline{IS} \cong \overline{CO}$).*
- *The midsegment of a trapezoid is the segment connecting the midpoints of the non-parallel sides.*
- *The midsegment of a trapezoid is parallel to the bases. Its length is the average of the lengths of the two bases.*

Frequently the bases of a trapezoid are designated by the variables b_1 and b_2 . Write an expression for the length of the midsegment in terms of b_1 and b_2 .

The length of the midsegment is given by $\frac{b_1 + b_2}{2}$.

9. Draw a triangle on a coordinate grid. Locate the midpoints of two of the sides. Draw the midsegment. Why is the midsegment parallel to the third side?

In the example shown, the slopes of the base and the midsegment are both $\frac{2}{3}$, and thus the base and the midsegment are parallel.



Compare the length of the midsegment to the length of the parallel side of the triangle.

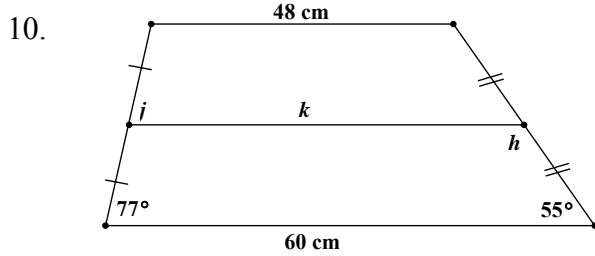
The length of the midsegment is half of the length of the parallel side.

Use the formula for the length of the midsegment of a trapezoid to justify this relationship.

Since b is the length of the parallel side, or base, of the triangle, then in the formula for the length of the midsegment, $\frac{b_1 + b_2}{2}$, $b_1 = 0$, $b_2 = b$. By substitution, the length of the

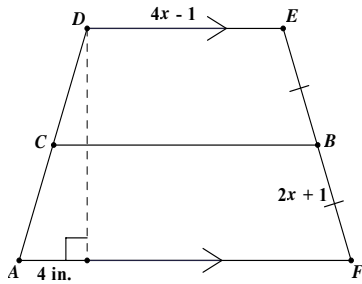
midsegment is $\frac{b}{2}$.

Application problems:



$$\begin{aligned} h &= 125^\circ \\ j &= 77^\circ \\ k &= 54 \text{ cm} \end{aligned}$$

11. The perimeter of isosceles trapezoid $ADEF$ is 218 in. \overline{BC} is the midsegment. Find AD .

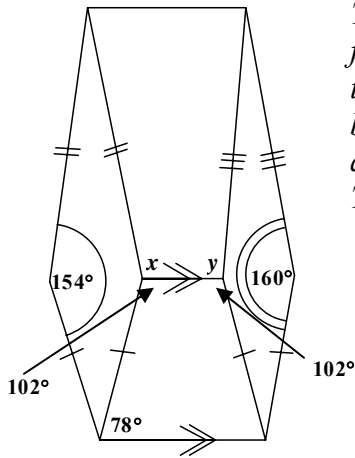


The non-parallel sides of the trapezoid are congruent. Each non-parallel side measures $2(2x + 1)$. The longer base is 8 in. longer than the shorter base.

$$\begin{aligned} \text{Perimeter of } ADEF &= 2(4x - 1) + 8 + 2[2(2x + 1)] \\ &= 8x - 2 + 8 + 8x + 4 \\ &= 16x + 10 \\ &= 218 \text{ in.} \\ 16x &= 208 \text{ in.} \\ x &= 13 \text{ in.} \end{aligned}$$

$$\begin{aligned} \text{Therefore, } AD &= 2(2x + 1) = 4x + 2 \\ &= 4(13) + 2 \\ &= 54 \text{ in.} \end{aligned}$$

12. In the two-dimensional figure, find the angle measure x and y . Explain.



The two quadrilaterals on the left and right sides of the figure are kites. The lower quadrilateral is a trapezoid, so the upper base angles are supplementary to the lower base angles. The measures of the upper trapezoid angles are both 102° .

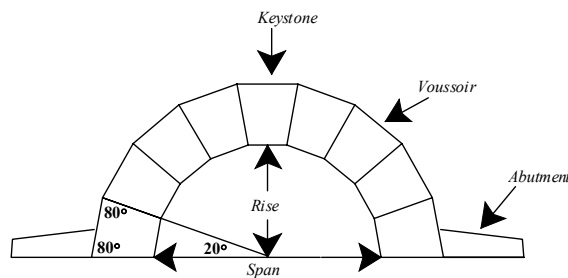
$$\text{Therefore } x = 360^\circ - 154^\circ - 102^\circ = 104^\circ;$$

$$y = 360^\circ - 160^\circ - 102^\circ = 98^\circ.$$

The following problems are taken from Serra, M. (2003). *Discovering Geometry: An Investigative Approach* (3rd ed.). Emeryville, CA: Key Curriculum Press, pp. 271, 283 with permission from Key Curriculum Press.

The Romans used the classical arch design in bridges, aqueducts, and buildings in the early centuries of the Common Era. The classical semicircular arch is really half of a regular polygon built with wedge-shaped blocks whose faces are isosceles trapezoids. Each block supports the blocks surrounding it.

13. The inner edge of the arch in the diagram is half of a regular 18-gon. Calculate the measures of all the angles in the nine isosceles trapezoids making up the arch.



Imagine that the isosceles trapezoids become isosceles triangles by regaining their truncated vertices. There are nine isosceles triangles, whose vertices meet at the center of a semicircle. The nine vertices each contribute 20° to the 180° at the center of the span. The sum of the base angles of each isosceles triangle is $180^\circ - 20^\circ = 160^\circ$. The trapezoid's base angles on the outer edge of the arch each measure 80° . The base angles on the inner edge of the arch are supplementary to the exterior base angle, and measure 100° each.

14. What is the measure of each angle in the isosceles trapezoid face of a voussoir in a 15-stone arch?

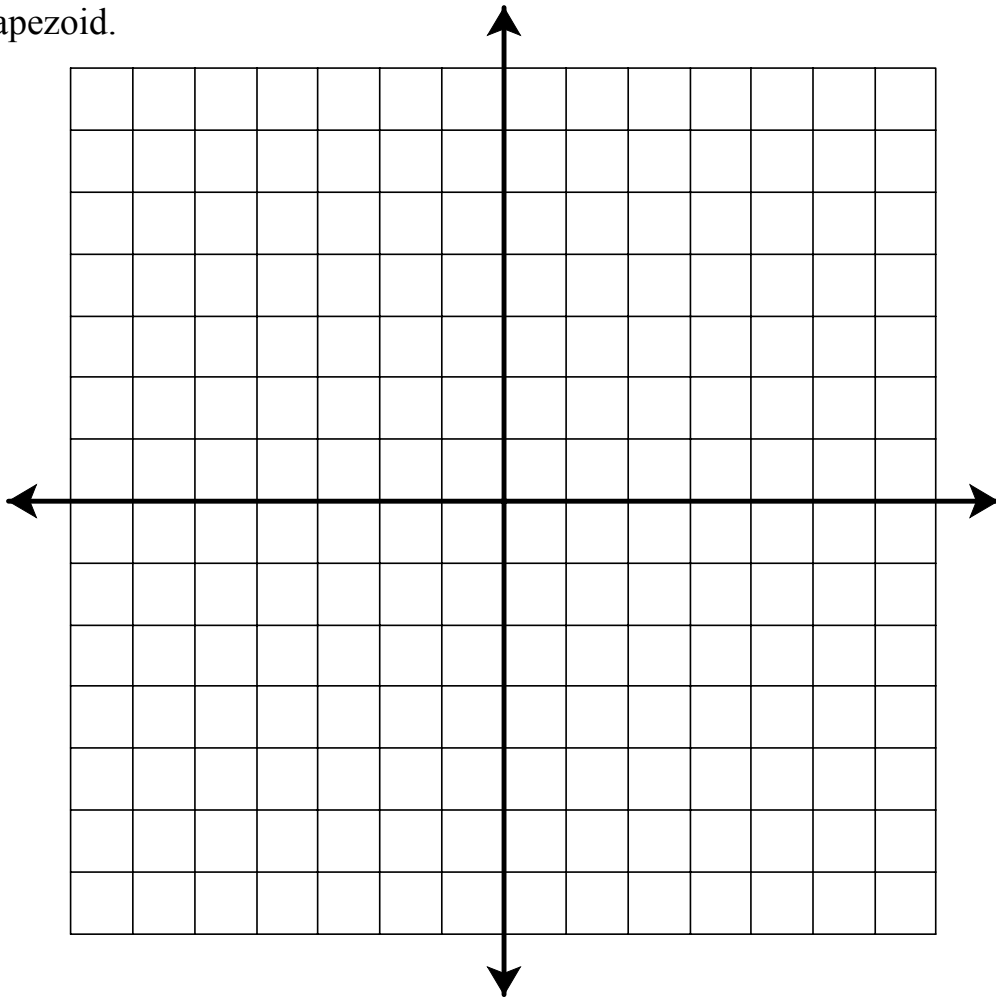
As in 13, the vertices of the isosceles triangles created by the 15 trapezoids span 180° . Each vertex spans 12° . The sum of the outer base angles in each trapezoid is $180^\circ - 12^\circ = 168^\circ$. Each outer base angle measures 84° . Each supplementary inner base angle measures 96° .

Remind participants to add the new terms isosceles trapezoid and midsegment to their glossaries.

Participants are performing at the van Hiele Relational Level as they develop properties of trapezoids and isosceles trapezoids using deductive reasoning rather than observation and measurement.

3. Work with your group to find the angle and side properties of trapezoids and isosceles trapezoids, using your knowledge of parallel lines, isosceles and scalene triangles. Be prepared to share and justify your properties during whole class discussion. Note that some books define a trapezoid as a quadrilateral with “at least one pair of parallel sides.” Others define a trapezoid as a quadrilateral with “only one pair of parallel sides”. This module will use the latter definition.

4. Locate the vertices of quadrilateral $ABCD$ on the coordinate plane at $A(-7, -3)$, $B(-3, 3)$, $C(1, 5)$, and $D(5, 3)$. Explain why $ABCD$ is a trapezoid.



5. Find the midpoints, M and L , of \overline{AB} and \overline{CD} respectively. \overline{ML} is called a midsegment.

6. How do the coordinates of a midpoint of a segment relate to the coordinates of its endpoints?

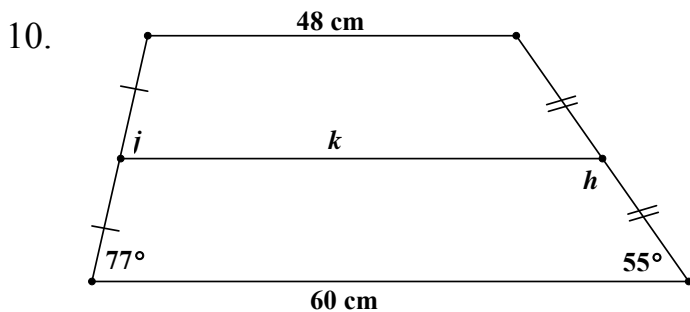
7. How does the slope of the midsegment relate to the slopes of the parallel sides of the trapezoid?

8. How does the length of the midsegment relate to the lengths of the parallel sides of the trapezoid?

9. Draw a triangle on a coordinate grid. Locate the midpoints of two of the sides. Draw the midsegment.
Why is the midsegment parallel to the third side?

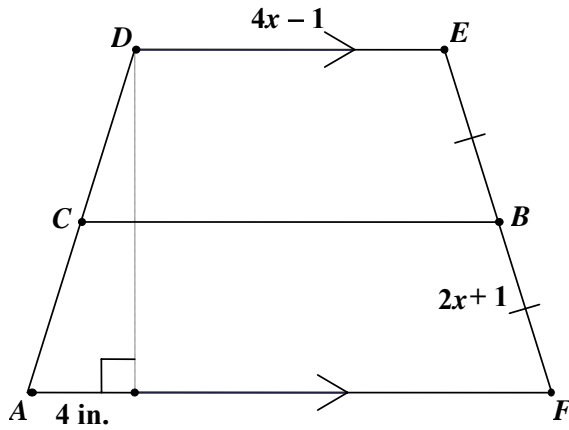
Compare the length of the midsegment to the length of the parallel side of the triangle.

Application problems:

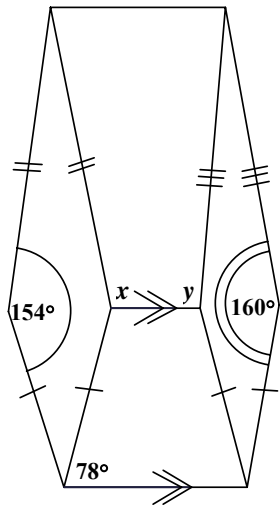


$h =$ _____ $^{\circ}$
 $j =$ _____ $^{\circ}$
 $k =$ _____ cm

11. The perimeter of isosceles trapezoid $ADEF$ is 218 in. \overline{BC} is the midsegment. Find AD .



12. In the two-dimensional figure below, find the angle measures x and y . Explain.

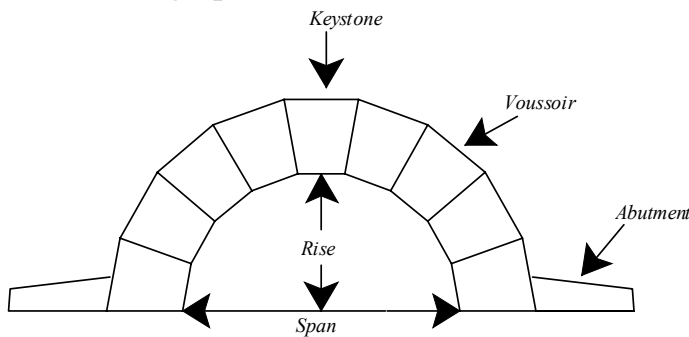


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13. The inner edge of the arch in the diagram is half of a regular 18-gon. Calculate the measures of all the angles in the nine isosceles trapezoids making up the arch.



14. What is the measure of each angle in the isosceles trapezoid face of a voussoir in a 15-stone arch?

Vesica Pisces

Overview: Using properties of the different quadrilaterals, participants determine the figures within the vesica pisces.

Objective: **TE_xES Mathematics Competencies**

III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.

III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).

V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.

V.018.F. The beginning teacher evaluates how well a mathematical model represents a real-world situation.

V.019.B. The beginning teacher understands how mathematics is used to model and solve problems in other disciplines (e.g., art, music, science, social science, business).

V.019.D. The beginning teacher communicates mathematical ideas using a variety of representations (e.g., numeric, verbal, graphical, pictorial, symbolic, concrete).

Geometry TEKS

b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.

b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

b.3.B. The student constructs and justifies statements about geometric figures and their properties.

b.3.E. The student uses deductive reasoning to prove a statement.

b.4. The student uses a variety of representations to describe geometric relationships and solve problems.

Background: Participants need to be familiar with triangle and quadrilateral properties.

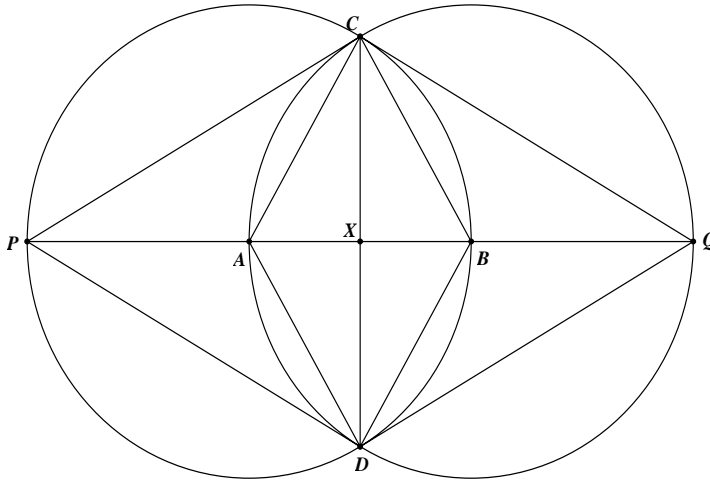
Materials: compass, easel paper, colored markers, centimeter ruler

New Terms: vesica pisces

Procedures:

Distribute the activity sheet.

Circles A and B pass through each others' centers and intersect at C and D .



Draw the following segments:

\overline{AB} , \overline{AC} , \overline{AD} , \overline{BC} , \overline{BD} , \overline{CD} .

\overline{AB} intersects circle A at P , and circle B at Q .

Draw

\overline{PQ} , \overline{PC} , \overline{PD} , \overline{QC} , \overline{QD} .

\overline{AB} intersects \overline{CD} at X .

In your group identify and classify triangles and quadrilaterals. Explain why the properties you have identified are true. For example, if a rhombus is identified on the basis of four congruent sides, explain why the four sides are congruent.

Allow participants 30 – 45 minutes to work together to complete the activity. Then ask individuals to share results, while one participant records on a sheet of easel paper.

This figure is called a vesica pisces. A vesica pisces is created by two identical intersecting circles, the circumference of one intersecting the center of the other.

All of the figures within the vesica pisces can be derived from the congruent radii \overline{AB} , \overline{AC} , \overline{AD} , \overline{BC} , and \overline{BD} . Some properties and figures follow. Many more can be discerned.

- \overline{ADBC} is a rhombus. ($\overline{AC} \cong \overline{AD} \cong \overline{BC} \cong \overline{BD}$.)
- $\overline{AB} \perp \overline{CD}$. (Diagonals of a rhombus are perpendicular to each other.)
- $AX = XB$. (Diagonals of a rhombus bisect each other.) $PA = BQ$ (Radii of congruent circles are congruent.) Therefore, $PA + AX = PX = XB + BQ = XQ$.
- \overline{PCQD} is a rhombus. (Diagonals of a rhombus are perpendicular bisectors of each other.)
- $\triangle ABC$ is equilateral. ($AB = AC = BC$.) Similarly, $\triangle ABD$ is equilateral.
- $m\angle PAC = 120^\circ$. ($m\angle CAB = 60^\circ$, since $\triangle ABC$ is equilateral.)
- $m\angle ACD = 30^\circ$. (Diagonal \overline{CD} of rhombus \overline{ACBD} bisects $\angle ACB$.)
- $m\angle PCA = 30^\circ$. ($\triangle PAC$ is isosceles since $PA = AC$; $m\angle PAC = 120^\circ$.)

- $m\angle PCD = 60^\circ$. ($m\angle ACD + m\angle PCA = m\angle PCD$.)
- $\triangle PCD$ is equilateral. ($\angle PCD \cong \angle PDC$, since the diagonals lie on the line of symmetry.)
- $m\angle DAC = 120^\circ = 2m\angle DPC$. (The measure of the arc is two times the measure of the inscribed angle intercepting the arc.)

Conclude the unit with the following discussion:

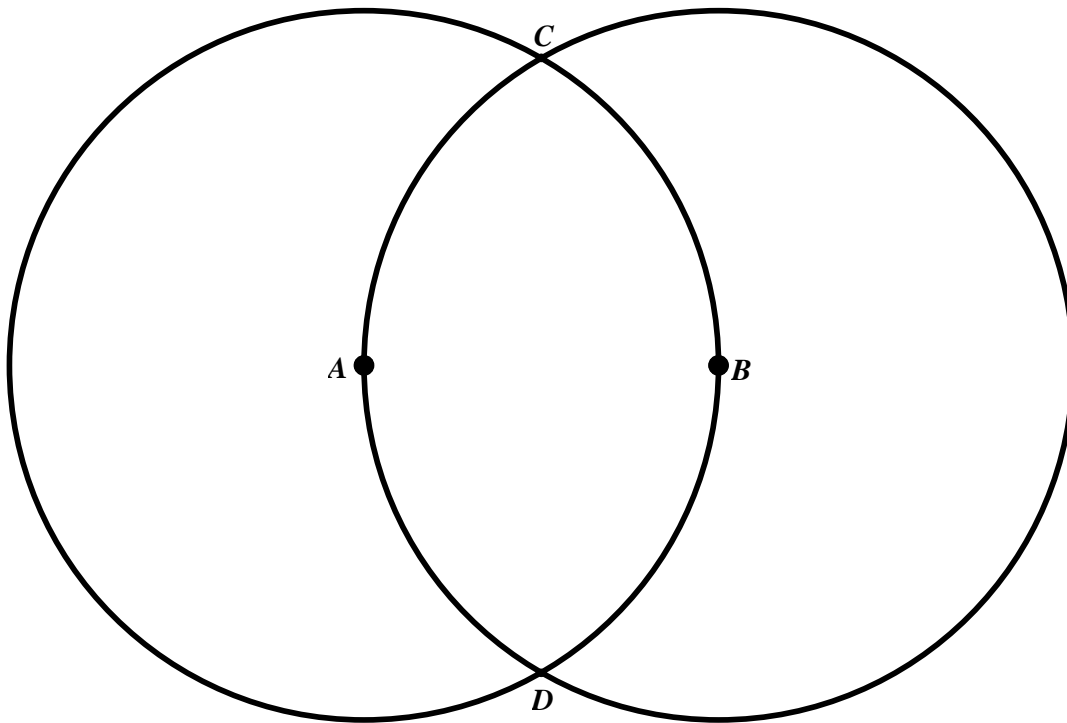
Explain to participants that the vesica pisces is the concept behind the traditional compass and straight edge construction of the perpendicular bisector of a segment, in this case, \overline{AB} . On a clean sheet of paper participants draw a segment and construct the perpendicular bisector using a compass. They then draw congruent circles with centers at the ends of the segments and radii equal to the lengths of the segments.

Participants are performing at the van Hiele Relational Level because they use logical justifications to explore relationships among properties of quadrilaterals.

Vesica Pisces



The vesica pisces on the lid of Chalice Well was designed by the excavator of Glastonbury Abbey, Frederick Bligh Bond, resident archaeologist of Glastonbury Abbey in the early 1900s. It was given to the Chalice Well as a thanks-offering for Peace in 1919, at the end of World War One, by friends and lovers of the Well and of Glastonbury.



Circles A and B pass through each others' centers and intersect at C and D .

Draw the following segments: \overline{AB} , \overline{AC} , \overline{AD} , \overline{BC} , \overline{BD} , \overline{CD} .

\overline{AB} intersects circle A at P and circle B at Q .

Draw \overline{PQ} , \overline{PC} , \overline{PD} , \overline{QC} , \overline{QD} .

\overline{AB} intersects \overline{CD} at X .

In your group identify and classify triangles and quadrilaterals using side lengths and angle measurements. Explain why the properties you have identified are true. For example, if a rhombus is identified on the basis of four congruent sides, explain why the four sides are congruent.

Exploring Prisms

Overview: Participants construct prisms by using the polygon as the base and the translation vectors from its vertices to construct a prism. This is done in a three-dimensional coordinate system. Participants explore and describe attributes of prisms.

Objectives: **TE_xES Mathematics Competencies**

II.004.A. The beginning teacher recognizes and extends patterns and relationships in data presented tables, sequences, or graphs.

III.011.B. The beginning teacher applies formulas for perimeter, area, surface area, and volume of geometric figures and shapes (e.g., polygons, pyramids, prisms, cylinders).

III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).

III.014.D. The beginning teacher applies transformations in the coordinate plane.

V.019.A. The beginning teacher recognizes and uses multiple representations of a mathematical concept (e.g., a point and its coordinates, the area of a circle as a quadratic function of the radius, probability as the ratio of two areas, area of a plane region as a definite integral).

V.019.C. The beginning teacher translates mathematical ideas between verbal and symbolic forms.

V.019.D. The beginning teacher communicates mathematical ideas using a variety of representations (e.g., numeric, verbal, graphical, pictorial, symbolic, concrete).

Geometry TEKS

b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

b.4. The student selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

d.2.A. The student uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures.

e.2.D. The student analyzes the characteristics of three-dimensional figures and their component parts.

e.3.A. The student uses congruence transformations to make conjectures and justify properties of geometric figures.

Background: Participants should be able to describe and perform a coordinate translation in two dimensions.

Materials: cardstock, scissors, floral wire, modeling clay, one-inch grid easel paper, Flash animation video 3-D.html, computer lab and/or computer with projector

New Terms: prism

Procedures:

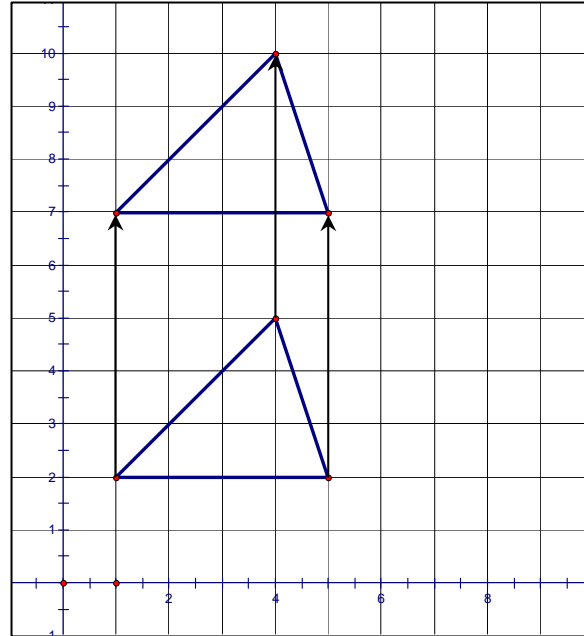
Assign a different polygon to each group of participants. After translating the polygons on grid paper according to the rule $(x, y) \rightarrow (x, y + 5)$, each group constructs a prism with the assigned polygon as its base. Suggestions follow:

- Group 1: triangle
- Group 2: square
- Group 3: rectangle (non-square)
- Group 4: regular pentagon
- Group 5: regular hexagon
- Group 6: regular octagon

Note: A template is provided for the regular pentagon, hexagon, and octagon in the activity pages.

1. On one-inch grid easel paper plot and label the coordinates of the vertices of your polygon. Connect the vertices to sketch the polygon.
In a two-dimensional coordinate plane, have participants plot ordered pairs that form the vertices of the assigned polygon. They then connect the points to sketch the polygon.
2. Translate the polygon in the xy -plane according to the rule $(x, y) \rightarrow (x, y + 5)$. Draw the image and the pre-image on the same coordinate grid. Draw the translation vectors.

Have participants translate the polygon according to the rule $(x, y) \rightarrow (x, y + 5)$. They should connect the vertices of the original polygon to the corresponding vertices of its image. A sample figure with translation vectors is shown on the next page.

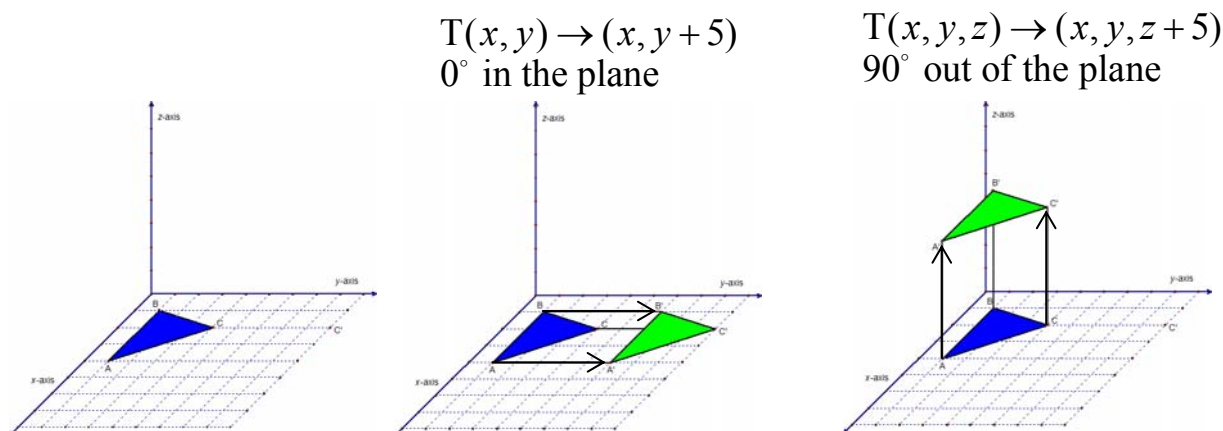


3. On a piece of cardstock, construct and cut out two polygons that are congruent to the one you plotted on the grid paper. Cut equal lengths of floral wire to use to connect the corresponding vertices of the two polygons. Use small balls of modeling clay to attach the ends of the floral wire to the corresponding vertices of the two polygons. Place one cardstock polygon over the original polygon and the other polygon over the image of the original polygon.

Note: Participants will need to keep their wire-frame prisms for use again in Exploring Pyramids and Cones.

4. What do the pieces of floral wire represent?
The pieces of floral wire represent the translation vectors. When the translation vectors are coplanar with the polygon, all parts of the figure are in the plane, in this case the xy -plane.
5. Keeping the floral wire connected to the polygons, translate the image out of the plane.
A sample translation is shown on the next page.

Use the Flash animation video 3-D.html to demonstrate the translation.



6. When the translation vectors no longer lie in the same plane as the two polygons, describe the resulting figure.

When the translation vectors are not in the xy -plane, the figure becomes a three-dimensional solid, a prism, generated by translating the polygon along these vectors. The original polygon and its image then become the bases of the prism and lie in parallel planes. The figures bounded by the florist wire and the edges of the bases are rectangles.

Remind participants to add the new term prism to their glossaries.

7. Complete the table below for your wire frame figure.
(Sample answer is shown for a square prism.)

Number of faces	Number of edges	Number of vertices	Height
6	12	8	5 in

8. Collect data from the other groups to complete the table below.
Sample data are shown in the table.

Name of Polygon	Number of faces	Number of edges	Number of vertices
<i>Triangle</i>	5	9	6
<i>Quadrilateral</i>	6	12	8
<i>Pentagon</i>	7	15	10
<i>Hexagon</i>	8	18	12
<i>Octagon</i>	10	24	16
<i>Decagon</i>	12	30	20

9. What patterns and relationships do you notice?

Answers will vary but may include the following:

- *There are more edges than faces*
- *The number of vertices on the prism is twice the number of vertices on the original polygon*
- *The number of edges on the prism is three times the number of edges on the original polygon*
- *The number of faces on the prism is equal to the number of edges on the original polygon plus two*

10. Use your table to discover a rule relating the number of faces, F , number of edges, E , and number of vertices, V , of a prism.

Answers will vary but may include:

- $V + F = E + 2$
- $V + F - 2 = E$
- $V - 2 = E - F$
- $V + F - E = 2$

This relationship, $V + F - E = 2$, describes the Euler characteristic or Euler number for convex polyhedra.

Participants are working at the Descriptive Level in describing the properties of a prism. They are using inductive reasoning throughout this activity.

Exploring Prisms

Your instructor will assign a polygon to your group. Record the name of your polygon below.

Polygon:

1. On one-inch grid easel paper plot and label the coordinates of the vertices of your polygon. Connect the vertices to sketch the polygon.
2. Translate your polygon in the xy -plane according to the rule $(x, y) \rightarrow (x, y+5)$. Draw the image and pre-image on the same coordinate grid. Draw the translation vectors.
3. On a piece of cardstock, construct and cut out two polygons that are congruent to the one you plotted on the grid paper. Cut equal lengths of floral wire to use to connect the corresponding vertices of your two polygons. Use small balls of modeling clay to attach the ends of the floral wire to the corresponding vertices of the two polygons. Place one cardstock polygon over the original polygon and the other polygon over the image of the original polygon.
4. What do the pieces of floral wire represent?
5. Keeping the floral wire connected to the polygons, translate the image out of the plane.
6. When the translation vectors no longer lie in the same plane as the two polygons, describe the resulting figure.
7. Complete the table below for your wire frame figure.

Number of faces	Number of edges	Number of vertices	Height

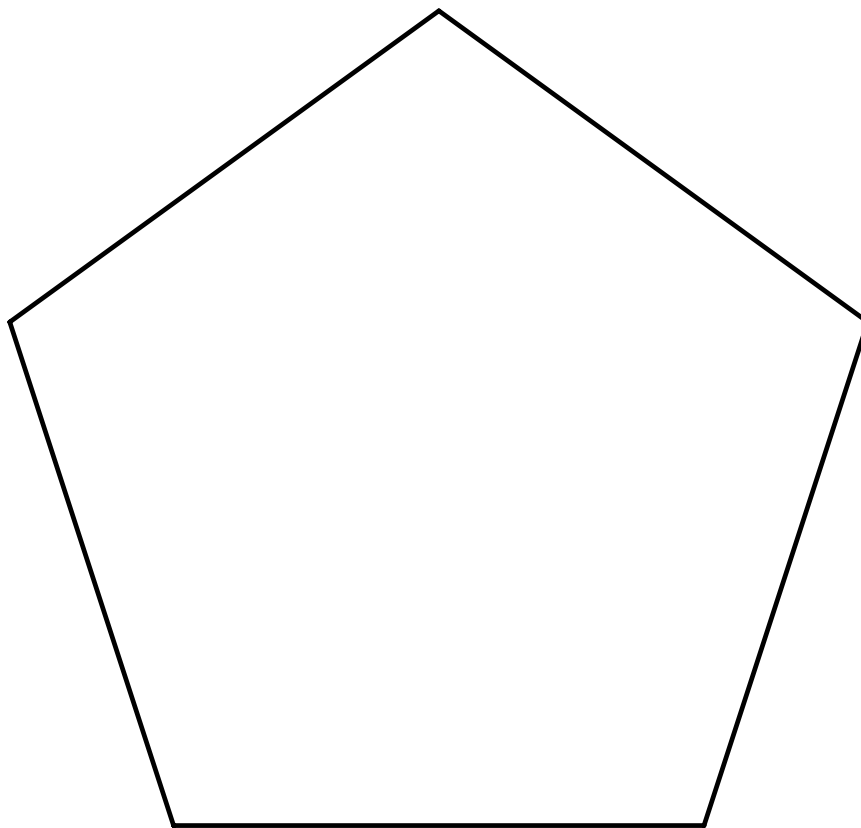
8. Collect data from the other groups to complete the table below.

Name of Polygon	Number of faces	Number of edges	Number of vertices

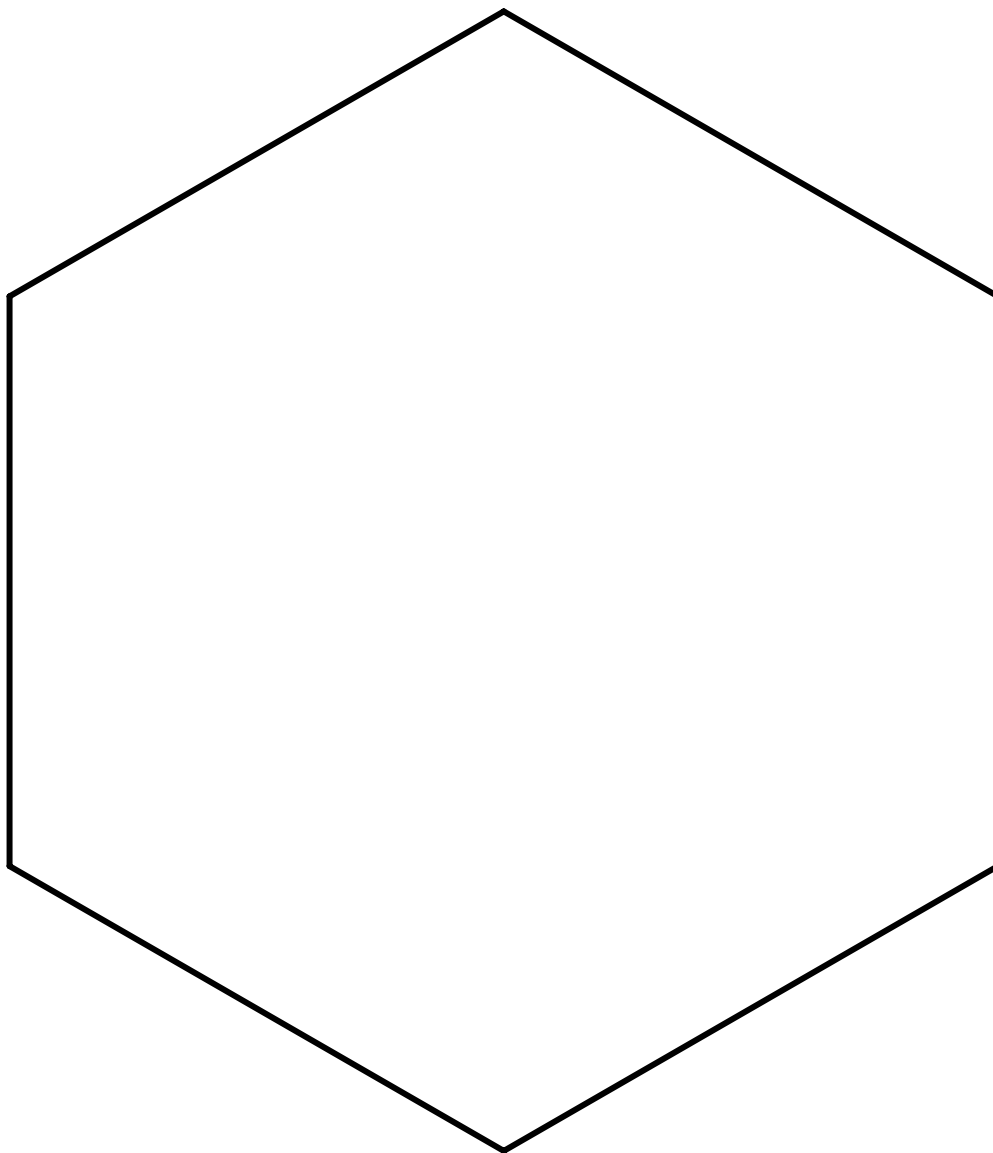
9. What patterns and relationships do you notice?

10. Use your table to discover a rule relating the number of faces, F , the number of edges, E , and the number of vertices, V , of a prism.

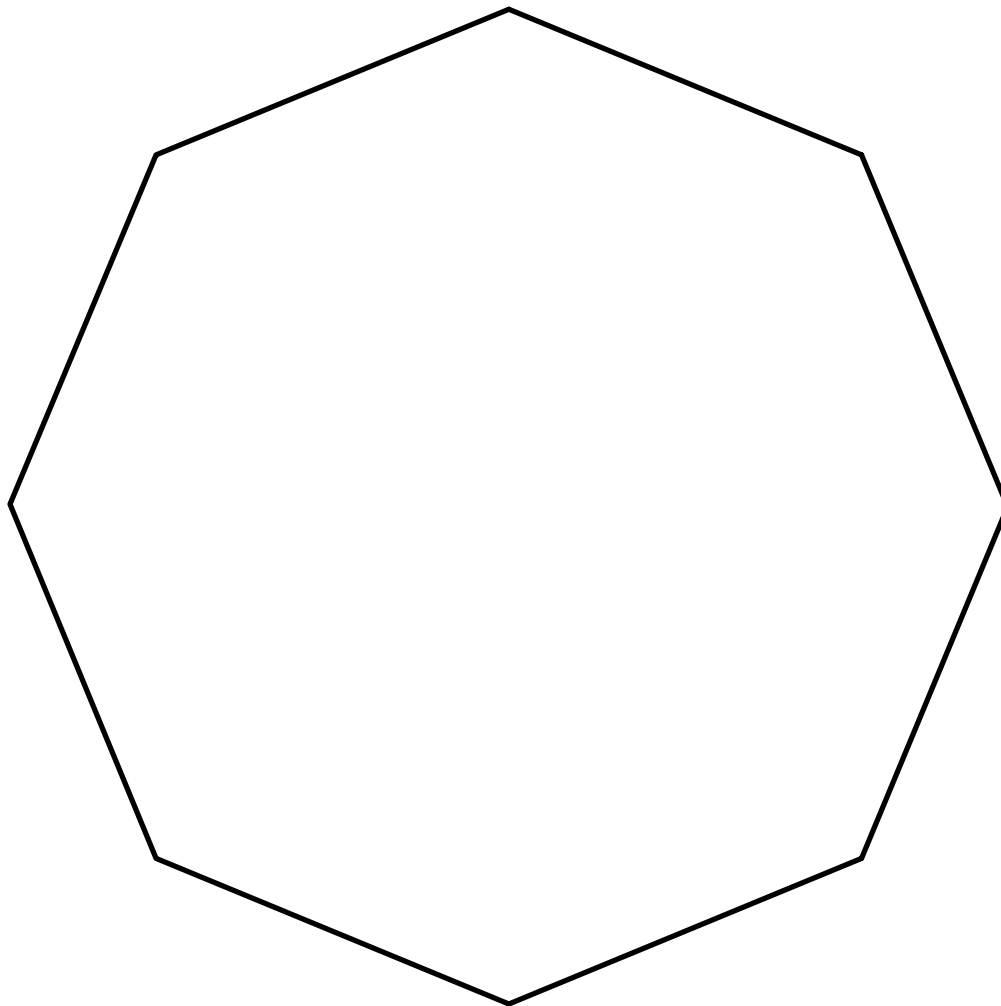
Template – Regular Pentagon



Template – Regular Hexagon



Template – Regular Octagon



References and Additional Resources

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