First Midterm Exam – Solutions
Math 102 – Spring 2008

1. **Problem 1 (10 pts)**
   Evaluate the integral $\int \frac{x}{x^4+1}\,dx$.

   The substitution $u = x^2$ leads to

   $$\int \frac{x}{x^4+1}\,dx = \frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\arctan u + C = \frac{1}{2}\arctan(x^2) + C$$

2. **Problem 2 (10 pts)**
   Evaluate the integral $\int x(\ln x)^2\,dx$.

   Use integration by parts with $(\ln x)^2 = u$ and $x = dv$:

   $$\begin{aligned} \int x(\ln x)^2\,dx &= \frac{1}{2}x^2(\ln x)^2 - \int \frac{x^2}{x}\ln x\,dx = \frac{1}{2}x^2(\ln x)^2 - \int x\ln x\,dx \\ &= \frac{1}{2}x^2(\ln x)^2 - \left(\frac{1}{2}x^2\ln x - \frac{1}{2}\int \frac{x^2}{x}\,dx\right) \\ &= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + C. \end{aligned}$$

3. **Problem 3 (10 pts)**
   Evaluate the integral $\int \sin^{10}\theta \cos^3\theta \, d\theta$.

Use $\cos^2\theta = 1 - \sin^2\theta$ and substitute $u = \sin\theta$ to get

$$\begin{aligned} \int \sin^{10}\theta \cos^3\theta \, d\theta &= \int \sin^{10}\theta(1-\sin^2\theta)\cos\theta \, d\theta \\ &= \int u^{10}(1-u^2)\, du = \frac{u^{11}}{11} - \frac{u^{13}}{13} + C \\ &= \frac{(\sin\theta)^{11}}{11} - \frac{(\sin\theta)^{13}}{13} + C \end{aligned}$$

4. **Problem 4 (10 points)**
   Evaluate the integral $\int \frac{1}{x^3 - x^2} \, dx$.

As $x^3 - x^2 = x^2(x-1)$ the partial fraction decomposition takes the form

$$\frac{1}{x^2(x-1)} = \frac{A}{x-1} + \frac{B}{x} + \frac{C}{x^2}.$$

This leads to $1 = (A+B)x^2 + (C-B)x - C$ and $C = -1$, $B = 1$, $A = -1$. Thus

$$\begin{aligned} \int \frac{1}{x^3 - x^2}\, dx &= -\int \frac{dx}{x-1} + \int \frac{dx}{x} - \int \frac{dx}{x^2} \\ &= -\ln|x-1| + \ln|x| + \frac{1}{x} + C \end{aligned}$$

5. **Problem 5 (10 points)**
Evaluate the integral $\int \frac{dx}{x^2\sqrt{x^2-9}}$. Express your final answer without the use of trigonometric functions.

We use $\sec^2\theta - 1 = \tan^2\theta$ and the substitution $x = 3\sec\theta$ to get

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2-9}} &= \frac{1}{9}\int \frac{\sec\theta\tan\theta}{\sec^2\theta\tan\theta}\,d\theta = \frac{1}{9}\int \cos\theta\,d\theta \\ &= \frac{1}{9}\sin\theta + C = \frac{1}{9}\sin(\sec^{-1}\frac{x}{3}) + C\end{aligned}$$

With the identity $\sin\theta = \sqrt{\sec^2\theta - 1}/\sec\theta$ we conclude

$$\int \frac{dx}{x^2\sqrt{x^2-9}} = \frac{1}{9}\frac{\sqrt{x^2-9}}{x} + C$$

6. **Problem 6 (10 points)**
Determine whether or not the improper integral converges. If it converges, find its value. If it diverges to $\pm\infty$, specify which one.

$$\int_0^1 \frac{1}{x(\ln x)^2}\,dx$$

$$\begin{aligned}\int_0^1 \frac{1}{x(\ln x)^2}\,dx &= \int_0^{\frac{1}{2}} \frac{1}{x(\ln x)^2}\,dx + \int_{\frac{1}{2}}^1 \frac{1}{x(\ln x)^2}\,dx \\ &= \lim_{s\to 0^+}\int_s^{\frac{1}{2}} \frac{1}{x(\ln x)^2}\,dx + \lim_{t\to 1^-}\int_{\frac{1}{2}}^t \frac{1}{x(\ln x)^2}\,dx \\ &= \lim_{s\to 0^+}\frac{-1}{\ln x}\Big]_s^{\frac{1}{2}} + \lim_{t\to 1^-}\frac{-1}{\ln x}\Big]_{\frac{1}{2}}^t = \frac{-1}{\ln(1/2)} + 0 + \infty + \frac{1}{\ln(1/2)},\end{aligned}$$

so the improper integral diverges to ∞.