Math 102 Fall 2008 Exam 1

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Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You have **fifty minutes**. Do all 5 problems. Please do all your work on the paper provided. You must show your work to receive full credit on a problem. An answer with no supporting work or explanation will receive little to no credit.
Please print you name clearly here.

Print name: ______________________________

Upon finishing please sign the pledge below:
On my honor I have neither given nor received any aid on this exam.

____

Grader's use only:

1. ________ /10

2. ________ /10

3. ________ /10

4. ________ /10

5. ________ /10

1. [10 points]

   Compute the integral

$$\int \ln x dx.$$

   We integrate by parts using $u = \ln x$ and $v' = 1$ so that $u' = 1/x$ and $v = x$. So we get

$$\int \ln x dx = x \ln x - \int \frac{1}{x} x dx = x \ln x - x + C.$$

2. [10 points]

Compute the integral

$$\int x^3\sqrt{1-x^2}dx.$$

One could first start by trying to simplify the integrand using integration by parts. But eventually we will have to make a trigonometric substitution so we might as well do it at the beginning.

We let $x = \sin t$ so that $dx = \cos t dt$. Then we get

$$\int x^3\sqrt{1-x^2}dx = \int \sin^3 t\sqrt{1-\sin^2 t}\cos t dt = \int \sin^3 t \cos^2 t dt.$$

Now we rewrite $\sin^3 t = \sin t \sin^2 t = \sin t(1-\cos^2 t)$ to get

$$\int \sin t(1-\cos^2 t)\cos^2 t dt = \int \sin t \cos^2 t dt - \int \sin^t \cos^4 t dt.$$

The reason for doing this last step is that now we can substitute $u = \cos t$ and $du = -\sin t dt$ to get

$$\int(-u^2)du - \int(-u^4)du = -\frac{u^3}{3} + \frac{u^4}{4} + C$$

Substituting back we get

$$-\frac{\cos^3 t}{3} + \frac{\cos^4 t}{4} + C = -\frac{(\cos\sin^{-1} x)^3}{3} + \frac{(\cos\sin^{-1} x)^4}{4} + C$$

Now $\cos\sin^{-1} x = \sqrt{1-x^2}$ so we get

$$-\frac{(1-x^2)^{3/2}}{3} + \frac{(1-x^2)^2}{4} + C.$$

3. [10 points]

Compute the integral

$$\int \frac{x^2-3}{(x+1)(x^2+2x+3)}dx.$$

Notice that we cannot factor x^2+2x+3 so we try to rewrite the integrant as

$$\frac{A}{x+1}+\frac{Bx+C}{x^2+x+3}.$$

So we get

$$\frac{x^2-3}{(x+1)(x^2+2x+3)}=\frac{A(x^2+2x+3)+(Bx+C)(x+1)}{(x+1)(x^2+2x+3)}$$

which gives (after expanding $(Bx+C)(x+1)=Bx^2+x(B+C)+C$ and collecting terms)

$$x^2-3=x^2(A+B)+x(2A+B+C)+(3A+C)$$

This means that $A+B=1$, $2A+B+C=0$ and $3A+C=-3$.

From the first of these we get $B=1-A$ and substituting into the second we get $2A+1-A+C=0$ or $A+C=-1$. Together with $3A+C=-3$ this gives $A=-1$ and $C=0$ so that $B=2$.

Now $\int \frac{-1}{x+1}dx=-\ln(1+x)+C$. On the other hand,

$$\int \frac{2x+2}{x^2+2x+3}dx=\ln(x^2+2x+3)+C.$$

Now $x^2+2x+3=(x+1)^2+2$ so that

$$\int \frac{2}{x^2+2x+3}dx=\int \frac{2}{(x+1)^2+2}dx=\int \frac{2}{u^2+2}du=\sqrt{2}\tan^{-1}(\frac{u}{\sqrt{2}})+C=\sqrt{2}\tan^{-1}(\frac{x+1}{\sqrt{2}})+C$$

where we substituted $u=x+1$.

Thus

$$\int \frac{2x}{x^2+2x+3}dx=\ln(x^2+2x+3)-\sqrt{2}\tan^{-1}(\frac{x+1}{\sqrt{2}})+C$$

and the final answer is

$$-\ln(1+x)+\ln(x^2+2x+3)-\sqrt{2}\tan^{-1}(\frac{x+1}{\sqrt{2}})+C.$$

4. [10 points] Determine whether or not the following integral converges. If it converges find its value. If it diverges to $\pm\infty$ indicate which one.

$$\int_0^1 \frac{\cos x}{\sin x} dx.$$

It is important here to realize that $\sin x = 0$ when $x = 0$ in the interval $[0, 1]$.

Now, substituting $u = \sin x$ we get

$$\int \frac{\cos x}{\sin x} dx = \int \frac{du}{u} = \ln u + C = \ln \sin x + C.$$

Notice that $\sin x > 0$ when $0 < x \leq 1$ so we get

$$\int_t^1 \frac{\cos x}{\sin x} dx = [\ln \sin(x)]t^1 = \ln \sin(1) - \ln \sin(t)$$

for $t > 0$. Now as $t \to 0^+$ then $\sin(t) \to 0^+$ so that $\ln \sin(t) \to -\infty$ so that

$$\lim_{t \to 0^+} \ln \sin(1) - \ln \sin(t) \to \infty$$

This means that the integral diverges to ∞.

5. [10 points]

Compute the integral

$$\int \frac{1}{\sin\theta} d\theta.$$

Multiply top and bottom by $\sin\theta$ to get

$$\int \frac{\sin\theta}{\sin^2\theta} d\theta = \int \frac{\sin\theta}{1-\cos^2\theta} d\theta.$$

Now we substitute $u = \cos\theta$ and $du = -\sin\theta d\theta$ to get

$$\int \frac{-du}{1-u^2} = -\int \frac{1}{1-u^2} du$$

Now $\frac{1}{1-u^2} = \frac{1/2}{1-u} + \frac{1/2}{1+u}$ so that we get

$$-(-\frac{1}{2}\ln(1-u) + \frac{1}{2}\ln(1+u))$$

and substituting back $u = \cos\theta$ we get

$$\frac{1}{2}\ln(1-\cos\theta) - \frac{1}{2}\ln(1+\cos\theta) + C.$$