

Some recommendations on using integration techniques

1. Substitution

- A substitution $u = g(x)$ works best, if the derivative $g'(x)$ appears as part of the integrand (but there are exceptions).
- A substitution can be used to use an integrand to a form which can be treated with other methods (e.g. one of the forms suitable for trigonometric substitution). Example: $\int x^5 \sqrt{x^3 - 1} dx$.

2. Integration by Parts

- If an integrand has two factors, which of them should one integrate and which one differentiate when using integration by part? As a rule: First choose log functions to differentiate. If no log functions are present, choose power functions.

Examples: In $\int x e^x dx$ choose $u = x$, $dv = e^x dx$, while in $\int x \ln x dx$ choose $u = \ln x$, $dv = x dx$ (it would not be good to choose $u = x$, $dv = \ln x dx$ here).

- In integrals of the form $\int e^x \sin x dx$ choose $u = \sin x$ and $dv = e^x dx$, do two integrations by part and then solve for the unknown integral.
- Example: $\int x e^x \sin x dx$. Start by finding $\int e^x \sin x dx$ and then choose $u = x$, $dv = e^x \sin x dx$ to integrate the original integral by parts.

3. Trigonometric Integrals

- Integrals of the form $\int \sin^m x \cos^n x dx$:
If m or n are odd, then use $\sin^2 x + \cos^2 x = 1$ on one of the odd-powered terms and then substitute $u = \sin x$ or $u = \cos x$.

Example: $\int \sin^5 x \cos^4 x dx = \int (1 - \cos^2 x)^2 \sin x \cos^3 x dx$. Substitute $u = \cos x$.

If m and n are both odd, then reduce the integrand to odd powered sines or cosines by using half-angle identities.

- Integrals of the form $\int \tan^m x \sec^n x dx$:
If m is odd, use the identity $\tan^2 x = \sec^2 x - 1$ to replace all but one of the tangent factors by secants, then substitute $u = \sec x$. If n is even, use $\sec^2 x = 1 + \tan^2$ and substitute $u = \tan x$.
Example: $\int \tan^3 x \sec^3 x dx = \int (\sec^2 x - 1) \sec^2 x \sec x \tan x dx = \int (u^4 - u^2) du$.

4. Partial Fractions

- If necessary, start by long division to turn an improper rational function into a proper rational function.
- Factor the denominator. If the denominator can be fully factored, use a “standard” partial fraction decomposition. If the denominator contains irreducible quadratic factors, then other methods have to be used (completing the square, substitution to reduce to simpler forms of the denominator, e.g. $x^2 + 1$, splitting numerator)

5. Trigonometric Substitutions

- All three types of trigonometric substitutions are based on trying to exploit $1 - \sin^2 x = \cos^2 x$ and its two consequences $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = 1 + \tan^2 x$. These identities directly suggest the proper substitutions to simplify expressions involving the terms $a^2 - x^2$, $a^2 + x^2$ or $x^2 - a^2$.
- Example: $\int \sqrt{2x - x^2} dx$.

6. Improper Integrals

- Start by splitting an improper integral by splitting it at all singularities, meaning at $+\infty$, $-\infty$ and all discontinuities of the integrand. Treat each term separately as an improper integral.
- When evaluating an improper integral, first find the indefinite integral in a calculation on the side. Then insert the antiderivative found into the correct boundaries for the improper integral. Do not try to directly substitute integration boundaries in an improper integral.
- Example: $\int_0^\infty \frac{1}{x^{1/2} + x^{3/2}} dx$