Math 102 Spring 2008: **Solutions: HW #10**

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1. section 10.8, #14

   The ratio test gives us (after dropping the $(-1)^n$ term because we take the absolute value)

   $$\frac{4^{n+1}x^{n+1}/(n+1)\ln(n+1)}{4^n x^n/n\ln n} = \frac{4xn\ln n}{(n+1)\ln(n+1)}$$

   Now as $n \to \infty$ we have that $\frac{n}{n+1} \to 1$ and

   $$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \lim_{n\to\infty} \frac{1/n}{1/(n+1)} = 1$$

   by L'Hôpital's rule. Thus the limit of $\frac{4xn\ln n}{(n+1)\ln(n+1)}$ is $4x$. By the ratio test this means the series converges for $-1/4 < x < 1/4$ and diverges if $|x| > 1/4$.

   Now when $x = 1/4$ the series converges by the alternating series test. When $x = -1/4$ the series is

   $$\sum \frac{1}{n\ln n}$$

   which diverges by the integral test. Namely, the integral of $\frac{1}{x\ln x}$ is $\ln\ln x$ which diverges.

2. section 10.8, #38

   The series for $(1+x)^{3/2}$ is

   $$1 + \frac{3}{2}x + \frac{3}{2}\frac{1}{2}\frac{x^2}{2!} + \frac{3}{2}\frac{1}{2}\frac{(-1)}{2}\frac{x^3}{3!} + \dots$$

   plugging in x^2 for x gives the power series for $(1+x^2)^{3/2}$.

   Now the radius of convergence of $(1+x)^{3/2}$ is 1 i.e. it convergence when $-1 < x < 1$ and diverges for $|x| > 1$. So $(1+x^2)^{3/2}$ converges for $-1 < x^2 < 1$ which is equivalent to $-1 < x < 1$ so the radius of convergence of $(1+x^2)^{3/2}$ is also 1.

3. section 10.8, #40

   We can write $\frac{1}{\sqrt{9+x^3}} = \frac{1}{3\sqrt{1+\frac{x^3}{9}}}$. Now $\frac{1}{3\sqrt{1+x}}$ has power series

   $$\frac{1}{3}(1 + \frac{1}{2}x + \frac{1}{2}\frac{-1}{2}\frac{x^2}{2!} + \frac{1}{2}\frac{-1}{2}\frac{-3}{2}\frac{x^3}{3!} + \dots)$$

   plugging in $\frac{x^3}{9}$ for x we get the power series we want.

Now the power series $\frac{1}{3\sqrt{1+x}}$ converges for $-1 < x < 1$ so has radius of convergence 1. Thus our power series converges for $-1 < \frac{x^3}{9} < 1$ or equivalently $-9 < x^3 < 9$ or equivalently $-9^{1/3} < x < 9^{1/3}$ so the radius of convergence is $9^{1/3}$.

4. section 10.8, #42

   We know from earlier computations in the chapter that

$$\tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots$$

   so that

$$x - \tan^{-1}(x) = \frac{1}{3}x^3 - \frac{1}{5}x^5 + \dots$$

   and dividing by x^3 we get

$$\frac{1}{3} - \frac{1}{5}x^2 + \frac{1}{7}x^4 - \dots$$

   Now the original power series for $\tan^{-1}(x)$ converges for $-1 < x < 1$ and subtracting x or dividing by x^3 does not change this fact. So the radius of convergence is 1.

5. section 10.8, #44

   We know that $\sin t$ has power series

$$t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots$$

   so that $\frac{\sin t}{t}$ has power series

$$1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \dots$$

   and integrating term by term we get

$$t - \frac{t^3}{3 \cdot 3!} + \frac{t^5}{5 \cdot 5!} - \dots$$

6. section 10.8, #48

   The power series for $\frac{1}{1-t^2}$ is

$$1 + t^2 + t^4 + t^6 + \dots$$

   so integrating term by term we get

$$t + \frac{t^3}{3} + \frac{t^5}{5} + \frac{t^7}{7} + \dots$$

7. section 10.8, #50

   We know that $\sum x^n = \frac{1}{1-x}$. Differentiating term by term we get that $\sum nx^{n-1}$ sums up to

$$\frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}$$

   Now differentiating again we get $\sum n(n-1)x^{n-2}$ sums up to

$$\frac{d}{dx}\frac{1}{(1-x)^2} = \frac{2}{(1-x)^3}$$

   Multiplying by x^2 we get that $\sum n(n-1)x^n$ sums up to

$$\frac{2x^2}{(1-x)^3}.$$

8. section 10.8, #52

   We know that $\sum nx^n = \frac{x}{(1-x)^2}$ so plugging in $x = 1/2$ we get

$$\sum \frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = 2$$

   Similarly, we know that

$$\sum n^2x^n = \frac{x(1+x)}{(1-x)^3}$$

   so plugging in $x = 1/3$ we get

$$\sum \frac{n^2}{3^n} = \frac{1/3(1+1/3)}{(1-1/3)^3} = \frac{4/9}{8/27} = 3/2$$

9. page 814 #38

   Using the ratio test we get

$$\frac{x^{n+1}/\ln(n+1)}{x^n/\ln n} = \frac{x\ln n}{\ln(n+1)}$$

   which converges to x as $n \to \infty$. Thus the series converges for $-1 < x < 1$ and diverges for $|x| > 1$.

   Now when $x = -1$ the series $\sum \frac{(-1)^n}{\ln n}$ converges by the alternating series test. When $x = 1$ we get $\sum \frac{1}{\ln n}$ which diverges by the comparison test with $\sum \frac{1}{n}$ (which in turn diverges by the integral test).

10. page 814 #40

    Again we use the ratio test to get

$$\frac{(1+\frac{1}{n+1})^{n+1}(x-1)^{n+1}}{(1+\frac{1}{n})^n(x-1)^n}$$

Now we know that $(1 + \frac{1}{n})^n$ tends to the constant e as $n \to \infty$. So the limit above as $n \to \infty$ is equal to $\frac{e}{e}(x-1) = x-1$. So the series converges if $-1 < x - 1 < 1$ or equivalently if $0 < x < 2$. It diverges if $x - 1 > 1$ or $x - 1 < -1$ or equivalently if $x > 2$ or $x < 0$.

Finally, if $x = 0$ or $x = 2$ the series diverges since the individual terms in the series do not tend to zero (their absolute value tends to e).