Happy fractals and an example of Whitney

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Dedicated to the memory of Juha Heinonen

Abstract

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Contents

1 Whitney's example

A famous example of Hassler Whitney shows that there is a nonempty compact connected set E in the plane and a continuously-differentiable real-valued function f on $\mathbf{R}^2$ such that $\nabla f(p) = 0$ for every $p \in E$ but f is not constant on E. This may seem rather strange, in view of the many standard results in calculus to the effect that functions with vanishing derivative should be constant. Of course, this would be trivial if E were not required to be connected.

More precisely, one can take E to be a continuous curve in the plane. If instead E were a continuously-differentiable curve, then it would follow from calculus that f is constant on E. Similarly, one can show that the same conclusion holds for a continuous curve of finite length.

If f were twice continuously-differentiable, then Sard's theorem would imply that $f(E)$ has Lebesgue measure 0 in the real line. Because E is connected and f is continuous, we also know that $f(E)$ is a connected set in $\mathbf{R}$. Hence $f(E)$ has only one element under these conditions, and so f is constant on E.

2 Whitney's extension theorem

Suppose that E is a compact set in the plane, and that f is a continuously-differentiable real-valued function on $\mathbf{R}^2$ with $\nabla f(p) = 0$ for every $p \in E$. Of course, $\nabla f(p) = 0$ is equivalent to

$$\lim_{x \to p} \frac{|f(x) - f(p)|}{|x - p|} = 0, \tag{2.1}$$

by the definition of differentiability. Using the compactness of E and the continuity of ∇f, one can show that (2.1) holds with uniform convergence for $p \in E$. This means that for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\frac{|f(x) - f(p)|}{|x - p|} < \epsilon \tag{2.2}$$

for every $x \in \mathbf{R}^2$ and $p \in E$ with $x \neq p$ and $|x - p| < \delta$.

Conversely, if f is a real-valued function on E such that (2.1) holds with uniform convergence on E, then Whitney's extension theorem implies that there is a continuously-differentiable function F on $\mathbf{R}^2$ such that $\nabla F(p) = 0$ and $F(p) = f(p)$ for each $p \in E$. Whitney's theorem also applies to functions whose gradient may not be 0 at every element of E, but this special case is easier to state and sufficient for the present purposes. To get an example as in the previous section, it is therefore enough to show that there is a nonconstant function f on E for which (2.1) holds with uniform convergence on E.

3 Sierpinski gaskets and carpets

Let T be an equilateral triangle in the plane. More precisely, this means the compact set in $\mathbf{R}^2$ consisting of the three line segments in the boundary of T as well as the interior of T. As usual, we can subdivide T into four smaller equilateral triangles, where the smaller triangles have disjoint interiors and sidelength equal to one-half the sidelength of T. At the first stage of the construction, we keep only the three smaller triangle in the corners of T. By repeating the process, we get nine equilateral triangles with disjoint interiors and sidelength equal to one-fourth the sidelength of T. At the nth stage of the construction, we get a compact set $E_n \subseteq \mathbf{R}^2$ which is the union of 3^n equilateral triangles with disjoint interiors and sidelength equal to 2^{-n} times the sidelength of T. By including the interiors of the triangles, we have that $E_{n+1} \subseteq E_n \subseteq T$ for each n, and

$$E = \bigcap_{n=1}^{\infty} E_n \tag{3.1}$$

is the Sierpinski gasket.

By construction, the Sierpinski gasket is a compact set in the plane with empty interior. It is easy to see that E is also connected, and it follows that E has topological dimension equal to 1. The Hausdorff dimension of E is $\log 3/\log 2$, and more precisely the Hausdorff measure of E is positive and finite in this dimension. If f is a continuously-differentiable function on $\mathbf{R}^2$ such that $\nabla f(p) = 0$ for every $p \in E$, then it is easy to see that f is constant on E. Indeed, the restriction of f to any line segment contained in E is constant, and there are a lot of line segments in E.

Similarly, let Q be a square in the plane, which is a compact set consisting of four line segments in the boundary as well as the interior. As before, Q can be subdivided into nine smaller squares with disjoint interiors and sidelength equal to one-third the sidelength of Q. At the first stage of the construction, we keep all of the smaller squares except for the one in the middle. Repeating the process, we get at the nth stage a compact set C_n consisting of 8^n squares with disjoint interiors and sidelength equal to 3^{-n} times the sidelength of Q. By including the interiors of the squares, we have that $C_{n+1} \subseteq C_n \subseteq Q$ for each n, and

$$C = \bigcap_{n=1}^{\infty} C_n \tag{3.2}$$

is the Sierpinski carpet. This is a compact, connected set in the plane, with empty interior, and topological dimension 1. The Hausdorff dimension of C is $\log 8/\log 3$, and the Hausdorff measure of C in this dimension is positive and finite. If f is a continuously-differentiable function on $\mathbf{R}^2$ such that $\nabla f(p) = 0$ for every $p \in C$, then it is easy to see that f is constant on C, because f is constant on every line segment in C.

4 Happy fractals

Let us say that a closed set $E \subseteq \mathbf{R}^n$ is a *happy fractal* if there is a real number $L \geq 1$ such that each $p, q \in E$ can be connected by a continuous path contained in E with length less than or equal to $L\,|p - q|$. Of course, this condition does not really require that E be fractal. Convex sets satisfy this condition with $L = 1$, and compact smooth submanifolds satisfy this condition for some L, which reflects the geometry of the embedding. One can show that Sierpinski gaskets and carpets are happy fractals, while snowflake curves and Cantor sets are not. More precisely, Cantor sets are totally disconnected and do not contain any nonconstant continuous paths at all, while snowflake curves do not contain nonconstant continuous paths of finite length.

The definition of a happy fractal can easily be extended to arbitrary metric spaces. It is customary to restrict one's attention to metric spaces that satisfy a doubling condition, as in [3, 4]. Roughly speaking, this puts a limit on the size of the metric space, which automatically holds for subsets of $\mathbf{R}^n$. This permits one to include examples related to Heisenberg groups and other nilpotent Lie groups, and sub-Riemannian geometry more broadly. In the context of ordinary Riemannian geometry, the distance between two points is defined to be the infimum of the lengths of the paths between them, and the doubling condition contains additional information.

Suppose that $E \subseteq \mathbf{R}^n$ is a happy fractal with constant L, and that f is a countinuously-differentiable function on $\mathbf{R}^n$. If $\nabla f(x) = 0$ for every $x \in E$, then it is easy to see that f is constant on E. As mentioned earlier, this works for any continuous path of finite length, and so it suffices to know that every pair of elements of E can be connected by such a path in E. If instead we ask that $|\nabla f(x)| \leq k$ for some $k \geq 0$ and every $x \in E$, then it follows that

$$|f(p) - f(q)| \leq k\,L\,|p - q| \tag{4.1}$$

for every $p, q \in E$. In particular, this implies that f is constant on E when

$k = 0$.

There are more intrinsic versions of the condition $|\nabla f| \le k$ on E, which also make sense on abstract metric spaces. This leads to more intrinsic versions of (4.1) as well. At any rate, the main point is that the restriction of a function to a curve determines a function on an interval in the real line.

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