An introduction to some aspects of functional analysis

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Abstract

These informal notes deal with some very basic objects in functional analysis, including norms and seminorms on vector spaces, bounded linear operators, and dual spaces of bounded linear functionals in particular.

Contents

1 Norms on vector spaces

Let V be a vector space over the real numbers $\mathbf{R}$ or the complex numbers $\mathbf{C}$. A *norm* on V is a nonnegative real-valued function $\|v\|$ defined for $v \in V$ such that

$$\|v\| = 0 \tag{1.1}$$

if and only if $v = 0$,

$$\|t\,v\| = |t|\,\|v\| \tag{1.2}$$

for every $v \in V$ and $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, and

$$\|v + w\| \le \|v\| + \|w\| \tag{1.3}$$

for every $v, w \in V$. Here $|t|$ is the absolute value of t when $t \in \mathbf{R}$, and the modulus of t when $t \in \mathbf{C}$. For example, if V is the one-dimensional vector space $\mathbf{R}$ or $\mathbf{C}$, then the absolute value or modulus defines a norm on V, and every norm on V is a positive multiple of this norm.

More generally, let n be a positive integer, and let V be the space $\mathbf{R}^n$ or $\mathbf{C}^n$ of n-tuples $v = (v_1, \ldots, v_n)$ of real or complex numbers. As usual, this is a vector space with respect to coordinatewise addition and scalar multiplication. The standard Euclidean norm on V is given by

$$\|v\| = \Big(\sum_{j=1}^{n} |v_j|^2\Big)^{1/2}. \tag{1.4}$$

It is well-known that this is a norm, although the triangle inequality (1.3) is not immediately obvious. A couple of ways to show the triangle inequality in this case will be reviewed in the next sections.

It is easy to see directly that

$$\|v\|_1 = \sum_{j=1}^{n} |v_j| \tag{1.5}$$

satisfies the triangle inequality. Hence $\|v\|_1$ defines a norm on $\mathbf{R}^n$, $\mathbf{C}^n$. Similarly,

$$\|v\|_\infty = \max(|v_1|, \ldots, |v_n|) \tag{1.6}$$

defines a norm on $\mathbf{R}^n$, $\mathbf{C}^n$. The standard Euclidean norm on $\mathbf{R}^n$, $\mathbf{C}^n$ is sometimes denoted $\|v\|_2$.

If $\|v\|$ is a norm on a real or complex vector space V, then

$$d(v, w) = \|v - w\| \tag{1.7}$$

defines a metric on V. For instance, the standard Euclidean metric on $\mathbf{R}^n$ or $\mathbf{C}^n$ is the same as the metric associated to the standard Euclidean norm in this way. The metrics on $\mathbf{R}^n$ or $\mathbf{C}^n$ associated to the norms $\|v\|_1$, $\|v\|_\infty$ determine the same topology as the standard Euclidean metric, and we shall say more about this soon. The main point is that $\|v\|_1$, $\|v\|_2$, and $\|v\|_\infty$ are *equivalent norms* on $\mathbf{R}^n$ and $\mathbf{C}^n$ for each positive integer n, in the sense that each is bounded by constant multiples of the others.

2 Convexity

Let V be a real or complex vector space. As usual, a set $E \subseteq V$ is said to be *convex* if for every $x, y \in E$ and real number t with $0 < t < 1$, we have that

$$t\,x + (1 - t)\,y \tag{2.1}$$

is also an element of E. If $\|v\|$ is a norm on V, then the closed unit ball

$$B_1 = \{v \in V : \|v\| \le 1\} \tag{2.2}$$

is a convex set in V. Similarly, the open unit ball

$$\{v \in V : \|v\| < 1\} \tag{2.3}$$

is also a convex set in V.

Conversely, suppose that $\|v\|$ is a nonnegative real-valued function on V which is positive when $v \ne 0$ and satisfies the homogeneity condition (1.2). If the corresponding closed unit ball B_1 is convex, then one can check that $\|v\|$ satisfies the triangle inequality, and is thus a norm. For if v, w are nonzero vectors in V, then

$$v' = \frac{v}{\|v\|}, \quad w' = \frac{w}{\|w\|} \tag{2.4}$$

have norm 1, and the convexity of B_1 implies that

$$\frac{\|v\|}{\|v\| + \|w\|}\, v' + \frac{\|w\|}{\|v\| + \|w\|}\, w' \tag{2.5}$$

has norm less than or equal to 1. Equivalently,

$$\frac{v + w}{\|v\| + \|w\|} \in B_1, \tag{2.6}$$

which implies (1.3).

Let n be a positive integer, let p be a positive real number, and consider

$$\|v\|_p = \Big(\sum_{j=1}^n |v_j|^p\Big)^{1/p} \tag{2.7}$$

for $v \in \mathbf{R}^n$ or $\mathbf{C}^n$. This is the same as the standard Euclidean norm when $p = 2$, and is easily seen to be a norm when $p = 1$, as in the previous section. Let us check that the closed unit ball associated to $\|v\|_p$ is convex when $p > 1$, so that $\|v\|_p$ defines a norm in this case. This does not work for $p < 1$, even when $n = 2$.

Suppose that $v, w \in \mathbf{R}^n$ or $\mathbf{C}^n$ satisfy $\|v\|_p, \|w\|_p \le 1$, which is the same as

$$\sum_{j=1}^n |v_j|^p, \ \sum_{j=1}^n |w_j|^p \le 1. \tag{2.8}$$

If $p \ge 1$ and $0 < t < 1$, then

$$|t\, v_j + (1-t)\, w_j|^p \le (t\,|v_j| + (1-t)\,|w_j|)^p \le t\,|v_j|^p + (1-t)\,|w_j|^p \tag{2.9}$$

for each j, because of the convexity of the function $f(x) = x^p$ on the positive real numbers. Summing over j, we get that

$$\sum_{j=1}^n |t\, v_j + (1-t)\, w_j|^p \le t \sum_{j=1}^n |v_j|^p + (1-t) \sum_{j=1}^n |w_j|^p \le 1. \tag{2.10}$$

This implies that

$$\|t\, v + (1-t)\, w\|_p \le 1, \tag{2.11}$$

as desired.

3 Inner product spaces

An *inner product* on a real or complex vector space V is a real or complex-valued function $\langle v, w\rangle$ defined for $v, w \in V$, as appropriate, which satisfies the following conditions. First, $\langle v, w\rangle$ is linear as a function of v for each $w \in V$. Second,

$$\langle w, v\rangle = \langle v, w\rangle \tag{3.1}$$

for every $v, w \in V$ in the real case, while

$$\langle w, v\rangle = \overline{\langle v, w\rangle} \tag{3.2}$$

in the complex case. Here $\overline{z}$ denotes the complex conjugate of $z \in \mathbf{C}$, defined by

$$\overline{z} = x - y\, i \tag{3.3}$$

when $z = x + y\,i$ and $x, y \in \mathbf{R}$. It follows from this symmetry condition that $\langle v, w \rangle$ is linear in w in the real case, and conjugate-linear in w in the complex case. It also follows that

$$\langle v, v \rangle \in \mathbf{R} \tag{3.4}$$

for every $v \in V$, even in the complex case. The third requirement of an inner product is that

$$\langle v, v \rangle > 0 \tag{3.5}$$

for every $v \in V$ with $v \neq 0$, and of course the inner product is equal to 0 when $v = 0$.

If $\langle v, w \rangle$ is an inner product on V, then the *Cauchy–Schwarz inequality* states that

$$|\langle v, w \rangle| \le \langle v, v \rangle^{1/2} \, \langle w, w \rangle^{1/2} \tag{3.6}$$

for every $v, w \in V$. Because of homogeneity, it suffices to show that

$$|\langle v, w \rangle| \le 1 \tag{3.7}$$

when

$$\langle v, v \rangle = \langle w, w \rangle = 1, \tag{3.8}$$

since the inequality is trivial when v or w is 0. To do this, one can use the fact that

$$\langle v + t\,w, v + t\,w \rangle \ge 0 \tag{3.9}$$

for every $t \in \mathbf{R}$ or $\mathbf{C}$, and expand the inner product to estimate $\langle v, w \rangle$.

The norm on V associated to the inner product is defined by

$$\|v\| = \langle v, v \rangle^{1/2}. \tag{3.10}$$

This clearly satisfies the positivity and homogeneity requirements of a norm. In order to verify the triangle inequality, one can use the Cauchy–Schwarz inequality to get that

$$\begin{aligned} \|v + w\|^2 &= \langle v + w, v + w \rangle \\ &= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle \\ &\le \|v\|^2 + 2\,\|v\|\,\|w\| + \|w\|^2 = (\|v\| + \|w\|)^2 \end{aligned} \tag{3.11}$$

for every $v, w \in V$.

The standard inner product on $\mathbf{R}^n$ is defined by

$$\langle v, w \rangle = \sum_{j=1}^{n} v_j \, w_j. \tag{3.12}$$

Similarly, the standard inner product on $\mathbf{C}^n$ is defined by

$$\langle v, w \rangle = \sum_{j=1}^{n} v_j \, \overline{w_j}. \tag{3.13}$$

The corresponding norms are the standard Euclidean norms.

4 A few simple estimates

Observe that

$$\|v\|_\infty \le \|v\|_p \tag{4.1}$$

for each $v \in \mathbf{R}^n$ or $\mathbf{C}^n$ and $p > 0$. If $0 < p < q < \infty$, then

$$\sum_{j=1}^n |v_j|^q \le \|v\|_\infty^{q-p} \sum_{j=1}^n |v_j|^p \tag{4.2}$$

for every $v \in \mathbf{R}^n$ or $\mathbf{C}^n$, and hence

$$\|v\|_q \le \|v\|_\infty^{1-(p/q)} \, \|v\|_p^{p/q} \le \|v\|_p. \tag{4.3}$$

Thus $\|v\|_p$ is monotone decreasing in p.

In the other direction,

$$\|v\|_p \le n^{1/p} \, \|v\|_\infty \tag{4.4}$$

for each $v \in \mathbf{R}^n$ or $\mathbf{C}^n$ and $p > 0$. If $0 < p < q < \infty$, then

$$\|v\|_p \le n^{(1/p)-(1/q)} \, \|v\|_q. \tag{4.5}$$

To see this, remember that $f(x) = x^r$ is a convex function of $x \ge 0$ when $r \ge 1$, so that

$$\Big(\frac{1}{n} \sum_{j=1}^n a_j\Big)^r \le \frac{1}{n} \sum_{j=1}^n a_j^r \tag{4.6}$$

for any nonnegative real numbers $a_1, \ldots, a_n$. Applying this with $r = q/p$, we get that

$$\Big(\frac{1}{n} \sum_{j=1}^n |v_j|^p\Big)^{q/p} \le \frac{1}{n} \sum_{j=1}^n |v_j|^q, \tag{4.7}$$

or

$$\Big(\frac{1}{n} \sum_{j=1}^n |v_j|^p\Big)^{1/p} \le \Big(\frac{1}{n} \sum_{j=1}^n |v_j|^q\Big)^{1/q}, \tag{4.8}$$

as desired.

The fact that

$$\|v\|_1 \le \|v\|_p \tag{4.9}$$

when $0 < p < 1$ and $n = 2$ can be re-expressed as

$$(a+b)^p \le a^p + b^p \tag{4.10}$$

for every $a, b \ge 0$. It follows that

$$\sum_{j=1}^n |v_j + w_j|^p \le \sum_{j=1}^n |v_j|^p + \sum_{j=1}^n |w_j|^p \tag{4.11}$$

for every $v, w \in \mathbf{R}^n$ or $\mathbf{C}^n$ when $0 < p < 1$, and hence

$$\|v + w\|_p^p \le \|v\|_p^p + \|w\|_p^p. \tag{4.12}$$

This is a natural substitute for the triangle inequality in this case.

5 Summable functions

Let E be a nonempty set, and let $f(x)$ be a nonnegative real-valued function on E. If E has only finitely many elements, then the sum

$$\sum_{x \in E} f(x) \tag{5.1}$$

is defined in the usual way. Otherwise, (5.1) is defined to be the supremum of

$$\sum_{x \in E_1} f(x) \tag{5.2}$$

over all finite subsets E_1 of E, which may be $+\infty$. If there is a finite upper bound for the subsums (5.2), so that (5.1) is finite, then we say that f is *summable* on E. Note that f is summable on E if there is a nonnegative summable function F on E such that $f(x) \le F(x)$ for every $x \in E$.

If f_1, f_2 are nonnegative functions on E, then

$$\sum_{x \in E} (f_1(x) + f_2(x)) = \sum_{x \in E} f_1(x) + \sum_{x \in E} f_2(x). \tag{5.3}$$

This is easy to see, with the standard convention that

$$a + \infty = \infty + a = \infty, \quad 0 \le a \le \infty. \tag{5.4}$$

In particular, $f_1 + f_2$ is summable on E when f_1 and f_2 are summable on E. Similarly, if f is a nonnegative function on E and $t \ge 0$, then

$$\sum_{x \in E} t\, f(x) = t \sum_{x \in E} f(x). \tag{5.5}$$

This uses the conventions that $t \cdot \infty = \infty$ when $t > 0$, and $0 \cdot \infty = 0$.

A real or complex-valued function f on E is said to be summable on E if $|f(x)|$ is summable on E. It follows from the previous remarks that the summable functions on E form a linear subspace of the vector space of real or complex-valued functions on E, with respect to pointwise addition and scalar multiplication. If f is a summable function on E and $\epsilon > 0$, then the set

$$E_\epsilon(f) = \{x \in E : |f(x)| \ge \epsilon\} \tag{5.6}$$

has only finitely many elements, and hence

$$E(f) = \{x \in E : f(x) \ne 0\} \tag{5.7}$$

has only finitely or countably many elements. More precisely,

$$|E_\epsilon(f)| \le \frac{1}{\epsilon} \sum_{x \in E} |f(x)|, \tag{5.8}$$

where $|E_\epsilon(f)|$ denotes the number of elements of $E_\epsilon(f)$.

If f is a real or complex-valued summable function on E, then the sum (5.1) can be defined in various equivalent ways. One way is to express f as a linear combination of nonnegative summable functions, and then use the definition of the sum in that case. Another way is to enumerate the elements of $E(f)$ in a sequence, and treat the sum as an infinite series. A basic property of the sum is that

$$\Big|\sum_{x \in E} f(x)\Big| \le \sum_{x \in E} |f(x)| \tag{5.9}$$

for any summable function f on E. As usual, the sum of a summable function is also linear in the function.

6 p-Summable functions

Let E be a nonempty set, and let p be a positive real number. A real or complex-valued function f on E is said to be *p-summable* if $|f(x)|^p$ is a summable function on E. Thus p-summability is the same as summability when $p = 1$. One can check that the spaces

$$\ell^p(E, \mathbf{R}), \quad \ell^p(E, \mathbf{C}) \tag{6.1}$$

of p-summable real and complex-valued functions on E are linear subspaces of the vector spaces of all such functions on E. This can be extended to $p = \infty$ by taking $\ell^\infty(E, \mathbf{R})$, $\ell^\infty(E, \mathbf{C})$ to be the spaces of bounded real and complex-valued functions on E.

If f is a p-summable function on E for some p, $0 < p < \infty$, then we put

$$\|f\|_p = \Big(\sum_{x \in E} |f(x)|^p\Big)^{1/p}. \tag{6.2}$$

Here the sum over E is defined to be the supremum over all finite subsums, as in the previous section. Similarly, if f is a bounded function on E, then we put

$$\|f\|_\infty = \sup\{|f(x)| : x \in E\}. \tag{6.3}$$

It is easy to see that

$$\|f_1 + f_2\|_p \le \|f_1\|_p + \|f_2\|_p \tag{6.4}$$

when $f_1, f_2 \in \ell^p(E, \mathbf{R})$ or $\ell^p(E, \mathbf{C})$ and $1 \le p \le \infty$, so that $\|f\|_p$ defines a norm on these spaces in this case. This is quite straightforward when $p = 1, \infty$, and can be obtained from the corresponding statement for finite sums when $1 < p < \infty$. If $0 < p < 1$ and f_1, f_2 are p-summable functions on E, then

$$\|f_1 + f_2\|_p^p \le \|f_1\|_p^p + \|f_2\|_p^p. \tag{6.5}$$

This can be derived using the same argument as for finite sums.

If f is p-summable, $0 < p < \infty$, then f is bounded, and

$$\|f\|_\infty \le \|f\|_p. \tag{6.6}$$

Moreover, f is q-summable for each $q > p$, and

$$\|f\|_q \le \|f\|_p. \tag{6.7}$$

This can be shown in the same way as for finite sums.

A function f on E is said to "vanish at infinity" if for each $\epsilon > 0$ there are only finitely many $x \in E$ such that $|f(x)| \ge \epsilon$. The spaces

$$c_0(E, \mathbf{R}), \quad c_0(E, \mathbf{C}) \tag{6.8}$$

of real and complex-valued functions vanishing at infinity on E are linear subspaces of the corresponding spaces of bounded functions on E. These are proper linear subspaces when E has infinitely many elements, and otherwise every function on E is bounded and vanishes at infinity when E has only finitely many elements. As in the previous section, p-summable functions vanish at infinity for each $0 < p < \infty$.

7 $p = 2$

It is well known that

$$a\, b \le \frac{a^2 + b^2}{2} \tag{7.1}$$

for every $a, b \ge 0$, since

$$0 \le (a - b)^2 = a^2 - 2\, a\, b + b^2. \tag{7.2}$$

If f_1, f_2 are square-summable functions on a nonempty set E, then it follows that

$$|f_1(x)|\, |f_2(x)| \le \frac{|f_1(x)|^2}{2} + \frac{|f_2(x)|^2}{2} \tag{7.3}$$

for every $x \in E$, and hence

$$\sum_{x \in E} |f_1(x)|\, |f_2(x)| \le \frac{1}{2} \sum_{x \in E} |f_1(x)|^2 + \frac{1}{2} \sum_{x \in E} |f_2(x)|^2 < \infty. \tag{7.4}$$

This implies that the product $f_1(x)\, f_2(x)$ is summable on E. Alternatively, one can get the same conclusion using the Cauchy–Schwarz inequality for finite sums to show that

$$\sum_{x \in E} |f_1(x)|\, |f_2(x)| \le \Big(\sum_{x \in E} |f_1(x)|^2\Big)^{1/2} \Big(\sum_{x \in E} |f_2(x)|^2\Big)^{1/2}. \tag{7.5}$$

At any rate, this permits us to define an inner product on $\ell^2(E, \mathbf{R})$ by

$$\langle f_1, f_2 \rangle = \sum_{x \in E} f_1(x)\, f_2(x), \tag{7.6}$$

and on $\ell^2(E, \mathbf{C})$ by

$$\langle f_1, f_2 \rangle = \sum_{x \in E} f_1(x) \, \overline{f_2(x)}. \tag{7.7}$$

These are basically the same as the standard inner products on $\mathbf{R}^n$ and $\mathbf{C}^n$ when E is a finite set with n elements. The norms associated to these inner products are the same as $\|f\|_2$ defined in the previous section.

8 Bounded continuous functions

Let X be a topological space, and let

$$\mathcal{C}_b(X, \mathbf{R}), \quad \mathcal{C}_b(X, \mathbf{C}) \tag{8.1}$$

be the spaces of bounded continuous real or complex-valued functions on X. As usual, these are vector spaces with respect to pointwise addition and scalar multiplication. If X is compact, then every continuous function on X is bounded, and we may drop the subscript b. The supremum norm of a bounded continuous function f is defined by

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}. \tag{8.2}$$

It is easy to see that this defines a norm on these vector spaces. Of course, constant functions on X are automatically bounded and continuous. If X is equipped with the discrete topology, then every function on X is continuous.

Locally compact Hausdorff spaces form a nice class of topological spaces. If X is such a space, $K \subseteq X$ is compact, and $V \subseteq X$ is an open set with $K \subseteq V$, then it is well known that there is a continuous real-valued function f on X such that

$$f(x) = 1 \tag{8.3}$$

when $x \in K$,

$$f(x) = 0 \tag{8.4}$$

when $x \in X \backslash V$, and

$$0 \le f(x) \le 1 \tag{8.5}$$

for every $x \in X$. In particular, there are plenty of bounded continuous functions on X in this case.

A continuous function f on X is said to "vanish at infinity" if for every $\epsilon > 0$ there is a compact set $K \subseteq X$ such that

$$|f(x)| < \epsilon \tag{8.6}$$

for every $x \in X \backslash K$. If X is equipped with the discrete topology, then the compact subsets of X are the same as the finite sets, and this definition reduces to the previous one in the context of summable functions. For any X, the space of real or complex-valued continuous functions on X that vanish at infinity may be denoted

$$\mathcal{C}_0(X, \mathbf{R}), \quad \mathcal{C}_0(X, \mathbf{C}). \tag{8.7}$$

It is easy to see that these are linear subspaces of the corresponding spaces of bounded continuous functions on X, since continuous functions are bounded on compact sets. If X is compact, then every continuous function on X vanishes at infinity trivially.

The *support* of a continuous function f on X is the closure of the set where $f \neq 0$, i.e.,

$$\operatorname{supp} f = \overline{\{x \in X : f(x) \neq 0\}}. \tag{8.8}$$

The spaces of real or complex-valued continuous functions on X with compact support may be denoted

$$\mathcal{C}_{00}(X, \mathbf{R}), \quad \mathcal{C}_{00}(X, \mathbf{C}). \tag{8.9}$$

These are linear subspaces of the corresponding spaces of continuous functions on X that vanish at infinity, and are the same as the spaces of all continuous functions on X when X is compact. There are a lot of continuous functions with compact support on locally compact Hausdorff spaces, as before.

9 Integral norms

Let f be a continuous real or complex-valued function on the closed unit interval $[0, 1]$ in the real line. For each positive real number p, put

$$\|f\|_p = \Big(\int_0^1 |f(x)|^p \, dx \Big)^{1/p}. \tag{9.1}$$

This obviously satisfies the triangle inequality when $p = 1$, and one can show that it also holds for $p > 1$ using the same argument as for finite sums. Thus $\|f\|_p$ defines a norm on the vector space of continuous functions on $[0, 1]$ when $p \geq 1$. If $0 < p < 1$, then

$$\|f_1 + f_2\|_p^p \leq \|f_1\|_p^p + \|f_2\|_p^p \tag{9.2}$$

for all continuous functions f_1, f_2 on $[0, 1]$. We also have that

$$\|f\|_p \leq \|f\|_\infty \tag{9.3}$$

for each continuous function f on $[0, 1]$ and every $p > 0$, where $\|f\|_\infty$ is the supremum norm of f, as in the previous section. If $0 < p < q < \infty$, then one can check that

$$\|f\|_p \leq \|f\|_q, \tag{9.4}$$

using convexity as in Section 4.

Now let f be a continuous real or complex-valued function on the real line with compact support. For each $p > 0$, consider

$$\|f\|_p = \Big(\int_{-\infty}^{\infty} |f(x)|^p \, dx \Big)^{1/p}. \tag{9.5}$$

More precisely, the integral here can be reduced to one on a bounded interval, since f has compact support. As before, this defines a norm on the vector space of continuous functions on $\mathbf{R}$ with compact support when $p \ge 1$, and satisfies the usual substitute for the triangle inequality when $0 < p < 1$. However, $\|f\|_p$ is normally neither monotone increasing nor decreasing in p in this case.

If $p = 2$, then these norms associated to suitable inner products. On the unit interval, these inner products are given by

$$\langle f_1, f_2 \rangle = \int_0^1 f_1(x)\, f_2(x)\, dx \tag{9.6}$$

in the real case, and

$$\langle f_1, f_2 \rangle = \int_0^1 f_1(x)\, \overline{f_2(x)}\, dx \tag{9.7}$$

in the complex case. The inner products on continuous functions with compact support on $\mathbf{R}$ are defined similarly.

Of course, there are analogous norms and inner products for other situations in which integration is defined. This includes sums over discrete sets as another special case.

10 Completeness

Remember that a metric space $(M, d(x, y))$ is said to be *complete* if every Cauchy sequence in M converges to an element of M. If M is complete and $N \subseteq M$, then N is complete with respect to the restriction of $d(x, y)$ to $x, y \in N$ if and only if N is a closed set in M.

Let V be a real or complex vector space equipped with a norm and hence a metric, as in Section 1. If V is complete with respect to this metric, then V is said to be a *Banach space*. If the norm is also associated to an inner product, then V is said to be a *Hilbert space*.

It is well known that the real line is complete with respect to the standard metric, and that consequently $\mathbf{R}^n$ and $\mathbf{C}^n$ are complete for each n. One can use this to show that the $\ell^p(E)$ spaces are complete with respect to the p-norm $\|f\|_p$ for every nonempty set E and $1 \le p \le \infty$. For if $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence of functions on E with respect to the p-norm for some p, then $\{f_j(x)\}_{j=1}^\infty$ is a Cauchy sequence of real or complex numbers for each $x \in E$. Hence $\{f_j\}_{j=1}^\infty$ converges pointwise to a function f on E, and one can use the fact that the norms $\|f_j\|_p$ are uniformly bounded in j to show that $f \in \ell^p(E)$. Using the Cauchy condition for $\{f_j\}_{j=1}^\infty$ in $\ell^p(E)$, one can also show that $\{f_j\}_{j=1}^\infty$ converges to f in $\ell^p(E)$. Similarly, $\ell^p(E)$ is complete when $0 < p < 1$, with respect to the metric

$$\|f_1 - f_2\|_p^p. \tag{10.1}$$

In addition, one can check that $c_0(E)$ is a closed subspace of $\ell^\infty(E)$, and is thus complete with respect to the supremum norm.

The space $\mathcal{C}_b(X)$ of bounded continuous functions on any topological space X is complete with respect to the supremum norm. As in the previous paragraph, one can first show that a Cauchy sequence in $\mathcal{C}_b(X)$ converges pointwise, and then use the Cauchy condition with respect to the supremum norm to check that the sequence converges uniformly, and that the limit is also bounded and continuous. The space $\mathcal{C}_0(X)$ of continuous functions that vanish at infinity on X is a closed subspace of $\mathcal{C}_b(X)$, and hence also complete with respect to the supremum norm. However, the space of continuous functions on the unit interval is not complete with respect to the integral norm $\|f\|_p$ for any p, $1 \le p < \infty$, nor with respect to the metric $\|f_1 - f_2\|_p^p$ when $0 < p < 1$. The completions of these spaces can be described in terms of the Lebesgue integral.

11 Bounded linear mappings

Let V and W be vector spaces, both real or both complex, and equipped with norms $\|\cdot\|_V$, $\|\cdot\|_W$, respectively. A linear mapping $T : V \to W$ is said to be *bounded* if there is an $A \ge 0$ such that

$$\|T(v)\|_W \le A\,\|v\|_V \tag{11.1}$$

for every $v \in V$. It is easy to see that every linear mapping is bounded when V is $\mathbf{R}^n$ or $\mathbf{C}^n$ for some positive integer n. If $V = W$ with the same norm, then the identity mapping $I(v) = v$ is bounded, with $A = 1$.

If T is a linear mapping from V into W that satisfies (11.1), then

$$\|T(v_1) - T(v_2)\|_W \le A\,\|v_1 - v_2\|_V \tag{11.2}$$

for every $v_1, v_2 \in V$, and it follows that $T : V \to W$ is uniformly continuous. Conversely, if a linear mapping $T : V \to W$ is continuous at the origin, then it is bounded. For if there is a $\delta > 0$ such that

$$\|T(v)\|_W < 1 \tag{11.3}$$

when $v \in V$ satisfies $\|v\|_V < \delta$, then (11.1) holds with $A = 1/\delta$.

A linear mapping from a real or complex vector space V into the real or complex numbers, as appropriate, is also known as a *linear functional* on V. A linear functional on a vector space V with a norm $\|\cdot\|_V$ is bounded if it is bounded with respect to the standard norm on $\mathbf{R}$ or $\mathbf{C}$.

Remember that the *kernel* of a linear mapping $T : V \to W$ is the linear subspace of V consisting of the $v \in V$ such that $T(v) = 0$. By continuity, the kernel of a bounded linear mapping is a closed subspace of V. Conversely, one can show that a linear functional on V is bounded when its kernel is closed.

12 Continuous extensions

Let $(M, d(x, y))$ and $(N, \rho(u, v))$ be metric spaces. If N is complete, then every uniformly continuous mapping from a dense subset of M into N has a unique extension to a uniformly continuous mapping from M into N. Ordinary continuity is sufficient for uniqueness of the extension, but uniform continuity is needed for its existence.

Now let V and W be vector spaces, both real or both complex, and equipped with norms. If W is complete, then every bounded linear mapping from a dense linear subspace of V into W has a unique extension to a bounded linear mapping from V into W. Bounded linear mappings are automatically uniformly continuous, as in the previous section, and the remaining point to check is that the extension is also linear in this case. One can use the same constant A as in (11.1) for the extension as for the initial mapping too.

For example, let E be an infinite set, and let V be the space of real or complex-valued summable functions on E. It is easy to see that the space of functions f on E such that $f(x) \neq 0$ for only finitely many $x \in E$ is dense in $\ell^1(E)$ with respect to the ℓ^1 norm. For each f in this subspace, the sum

$$\sum_{x \in E} f(x) \tag{12.1}$$

can be defined in the obvious way, and satisfies

$$\Big|\sum_{x \in E} f(x)\Big| \le \sum_{x \in E} |f(x)| = \|f\|_1. \tag{12.2}$$

Hence there is a unique extension of (12.1) to a bounded linear functional on $\ell^1(E)$. This gives another approach to the definition of the sum of a summable function, as in Section 5.

13 Bounded linear mappings, 2

Let V and W be vector spaces, both real or both complex. The space $\mathcal{L}(V, W)$ of linear mappings from V into W is also a vector space in a natural way, since the sum of two linear mappings from V into W is also a linear mapping, as is the product of a linear mapping with a real or complex number, as appropriate. If V and W are equipped with norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively, then one can check that the space $\mathcal{BL}(V, W)$ of bounded linear mappings from V into W is a linear subspace of $\mathcal{L}(V, W)$.

The *operator norm* of a bounded linear mapping $T : V \to W$ is defined by

$$\|T\|_{op} = \sup\{\|T(v)\|_W : v \in V,\ \|v\|_V \le 1\}. \tag{13.1}$$

Equivalently,

$$\|T(v)\|_W \le \|T\|_{op}\, \|v\|_V \tag{13.2}$$

for every $v \in V$, and $\|T\|_{op}$ is the smallest number with this property, which is to say the smallest value of A for which (11.1) holds. It is easy to see that the operator norm defines a norm on the vector space $\mathcal{BL}(V, W)$. For example, if $V = W$ with the same norm, then the identity mapping $I(v) = v$ has operator norm equal to 1.

Suppose that V_1, V_2, and V_3 are vector spaces, all real or all complex, and equipped with norms. If $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ are bounded linear mappings, then their composition $T_2 \circ T_1 : V_1 \to V_3$, defined by

$$(T_2 \circ T_1)(v) = T_2(T_1(v)), \ v \in V, \tag{13.3}$$

is also a bounded linear mapping. Moreover,

$$\|T_2 \circ T_1\|_{op,13} \le \|T_1\|_{op,12} \, \|T_2\|_{op,23}, \tag{13.4}$$

where the subscripts indicate which spaces are involved in the operator norms.

If W is complete, then one can show that the space of bounded linear mappings from V into W is complete with respect to the operator norm. For if $\{T_j\}_{j=1}^\infty$ is a Cauchy sequence in $\mathcal{BL}(V, W)$, then $\{T_j(v)\}_{j=1}^\infty$ is a Cauchy sequence in W for each $v \in V$, and hence converges in W. The limit determines a linear mapping $T : V \to W$, and one can use the boundedness of the operator norms of the T_j's to show that T is bounded too. Using the Cauchy condition again, one can show that $\{T_j\}_{j=1}^\infty$ converges to T in the operator norm, as desired.

14 Bounded linear functionals

Let V be a real or complex vector space equipped with a norm $\| \cdot \|$. As in Section 11, a bounded linear functional on V is a bounded linear mapping from V into the real or complex numbers, as appropriate. The dual space V^* of bounded linear functionals is thus the same as $\mathcal{BL}(V, \mathbf{R})$ or $\mathcal{BL}(V, \mathbf{C})$, and the dual norm $\| \cdot \|_*$ on V^* is defined to be the corresponding operator norm, using the standard norm on $\mathbf{R}$ or $\mathbf{C}$. In particular, V^* is automatically complete with respect to the dual norm, as in the previous section.

Suppose that the norm $\|v\|$ on V is associated to an inner product $\langle v, w \rangle$. Each $w \in V$ determines a linear functional λ_w on V, defined by

$$\lambda_w(v) = \langle v, w \rangle. \tag{14.1}$$

The Cauchy–Schwarz inequality implies that λ_w is a bounded linear functional on V, and that

$$\|\lambda_w\|_* = \|w\|. \tag{14.2}$$

If V has finite dimension, then every linear functional on V is of the form λ_w for some $w \in V$. If V is a Hilbert space, then every bounded linear functional on V is of this form.

To see this, let λ be a bounded linear functional on V, which we may as well suppose is not identically equal to 0. Thus $\lambda(z) = 1$ for some $z \in V$, and we can put

$$\rho = \inf\{\|z\| : z \in V,\ \lambda(z) = 1\}. \tag{14.3}$$

Let $\{z_j\}_{j=1}^\infty$ be a sequence of elements of V such that $\lambda(z_j) = 1$ for each j and

$$\lim_{j \to \infty} \|z_j\| = \rho. \tag{14.4}$$

We would like to show that $\{z_j\}_{j=1}^\infty$ is a Cauchy sequence in V, and hence converges when V is a Hilbert space. Because the norm is associated to an inner product, we have the parallelogram law

$$\Big\|\frac{v+w}{2}\Big\|^2 + \Big\|\frac{v-w}{2}\Big\|^2 = \frac{\|v\|^2}{2} + \frac{\|w\|^2}{2} \tag{14.5}$$

for every $v, w \in V$. We want to apply this to terms of the sequence in order to show that

$$\|z_j - z_l\| \to 0 \text{ as } j, l \to \infty. \tag{14.6}$$

For each $j, l \geq 1$, $\lambda(z_j) = \lambda(z_l) = 1$, which implies that

$$\lambda\Big(\frac{z_j + z_l}{2}\Big) = 1, \tag{14.7}$$

and hence

$$\Big\|\frac{z_j + z_l}{2}\Big\| \geq \rho. \tag{14.8}$$

Using the parallelogram law, this implies in turn that

$$\rho^2 + \Big\|\frac{z_j - z_l}{2}\Big\|^2 \leq \frac{\|z_j\|^2}{2} + \frac{\|z_l\|^2}{2}. \tag{14.9}$$

It follows that $\{z_j\}_{j=1}^\infty$ is a Cauchy sequence, since $\|z_j\|, \|z_l\| \to \rho$ as $j, l \to \infty$.

Assuming that V is a Hilbert space, we get that $\{z_j\}_{j=1}^\infty$ converges to an element z of V. By construction, $\lambda(z) = 1$, and $\|z\| = \rho$ is minimal among vectors with this property. If $v \in V$ is in the kernel of λ, so that $\lambda(v) = 0$, then v is orthogonal to z, which is to say that

$$\langle v, z \rangle = 0. \tag{14.10}$$

This is because

$$\lambda(z + t\,v) = 1 \tag{14.11}$$

for every $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, and hence

$$\|z + t\,v\| \geq \|z\| \tag{14.12}$$

by minimality, which implies orthogonality, as in calculus. If $w = z/\|z\|^2$, then it follows that

$$\lambda(v) = \langle v, w \rangle = \lambda_w(v) \tag{14.13}$$

for every $v \in V$, as desired.

15 $\ell^1(E)^*$

Let E be a nonempty set, and let h be a bounded real or complex-valued function on E. If f is a summable function on E, then $f\,h$ is a summable function on E, and we can put

$$\lambda_h(f) = \sum_{x \in E} f(x)\,h(x). \tag{15.1}$$

This defines a bounded linear functional on $\ell^1(E)$, since

$$|\lambda_h(f)| \le \sum_{x \in E} |f(x)|\,|h(x)| \le \|h\|_\infty\,\|f\|_1. \tag{15.2}$$

For each $y \in E$, let δ_y be the function on E defined by $\delta_y(x) = 0$ when $x \ne y$ and $\delta_y(y) = 1$. Thus $\|\delta_y\|_1 = 1$ and

$$\lambda_h(\delta_y) = h(y) \tag{15.3}$$

for each $y \in E$, and one can use this to show that

$$\|\lambda_h\|_{1,*} = \|h\|_\infty, \tag{15.4}$$

where $\|\cdot\|_{1,*}$ denotes the dual norm with respect to the norm $\|f\|_1$ on $\ell^1(E)$.

Conversely, suppose that λ is a bounded linear functional on $\ell^1(E)$. For each $y \in E$, put

$$h(y) = \lambda(\delta_y). \tag{15.5}$$

Thus

$$|h(y)| \le \|\lambda\|_{1,*} \tag{15.6}$$

for each $y \in E$, since $\|\delta_y\|_1 = 1$. Equivalently, h is bounded on E, and

$$\|h\|_\infty \le \|\lambda\|_{1,*}. \tag{15.7}$$

It remains to show that $\lambda = \lambda_h$ as linear functionals on $\ell^1(E)$.

More precisely, we want to show that for each $f \in \ell^1(E)$,

$$\lambda(f) = \lambda_h(f) \tag{15.8}$$

If $f(x) \ne 0$ for only finitely many $x \in E$, then f is a linear combination of finitely many δ_y's, and this holds by linearity. Because these functions are dense in $\ell^1(E)$, the same conclusion holds for every $f \in \ell^1(E)$, by continuity.

16 $c_0(E)^*$

Let E be a nonempty set, and let h be a summable real or complex-valued function on E. If f is a bounded function on E, then $f\,h$ is a summable function on E, and we can put

$$\lambda_h(f) = \sum_{x \in E} f(x)\,h(x). \tag{16.1}$$

This defines a bounded linear functional on $\ell^\infty(E)$, since

$$|\lambda_h(f)| \le \sum_{x \in E} |f(x)|\,|h(x)| \le \|h\|_1\,\|f\|_\infty. \tag{16.2}$$

If we choose f so that $\|f\|_\infty = 1$ and

$$f(x)\,h(x) = |h(x)| \tag{16.3}$$

for each $x \in E$, then

$$\lambda_h(f) = \|h\|_1, \tag{16.4}$$

at it follows the dual norm of λ_h on $\ell^\infty(E)$ is equal to $\|h\|_1$.

We can also restrict λ_h to a bounded linear functional on $c_0(E)$ with dual norm less than or equal to $\|h\|_1$. If $E_1 \subseteq E$ has only finitely many elements, then we can choose a function f on E such that $f(x) = 0$ on $E \backslash E_1$, $\|f\|_\infty = 1$, and (16.3) holds on E_1. Thus $f \in c_0(E)$ and

$$\lambda_h(f) = \sum_{x \in E_1} |h(x)|, \tag{16.5}$$

and one can use this to show that the dual norm of the restriction of λ_h to $c_0(E)$ is still equal to $\|h\|_1$, by taking the supremum over all finite subsets E_1 of E.

Now let λ be any bounded linear functional on $c_0(E)$. As in the previous section, we can define a function h on E by

$$h(y) = \lambda(\delta_y). \tag{16.6}$$

By construction,

$$\lambda(f) = \sum_{x \in E} f(x)\,h(x) \tag{16.7}$$

when $f(x) \ne 0$ for only finitely many $x \in E$. If $E_1 \subseteq E$ has only finitely many elements, then we can choose f as in the preceding paragraph to get that

$$\sum_{x \in E_1} |h(x)| \tag{16.8}$$

is less than or equal to the dual norm of λ on $c_0(E)$. It follows that h is a summable function on E, and that $\|h\|_1$ is less than or equal to the dual norm of λ on $c_0(E)$.

It remains to check that

$$\lambda(f) = \lambda_h(f) \tag{16.9}$$

for every $f \in c_0(E)$. We already know that this holds when $f(x) \ne 0$ for only finitely many $x \in E$, and it holds for every $f \in c_0(E)$ by continuity, since these functions are dense in $c_0(E)$.

17 Hölder's inequality

Let $1 < p < \infty$ be given, and let $1 < q < \infty$ be the conjugate exponent, characterized by

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{17.1}$$

If a and b are nonnegative real numbers, then

$$a\,b \le \frac{a^p}{p} + \frac{b^q}{q}. \tag{17.2}$$

This can be seen as a consequence of the convexity of the exponential function, and the $p = q = 2$ case is elementary, as in Section 7.

Suppose that f, h are real or complex-valued functions on a nonempty set E. As in the previous paragraph,

$$|f(x)|\,|h(x)| \le \frac{|f(x)|^p}{p} + \frac{|h(x)|^q}{q} \tag{17.3}$$

for every $x \in E$. If f is p-summable and h is q-summable, then it follows that $f\,h$ is summable on E, with

$$\sum_{x \in E} |f(x)|\,|h(x)| \le \frac{1}{p} \sum_{x \in E} |f(x)|^p + \frac{1}{q} \sum_{x \in E} |h(x)|^q. \tag{17.4}$$

Equivalently,

$$\|f\,h\|_1 \le \frac{\|f\|_p^p}{p} + \frac{\|h\|_q^q}{q}. \tag{17.5}$$

If $\|f\|_p = \|h\|_q = 1$, then we get that

$$\|f\,h\|_1 \le 1. \tag{17.6}$$

This implies for any $f \in \ell^p(E)$ and $h \in \ell^q(E)$ that

$$\|f\,h\|_1 \le \|f\|_p\,\|h\|_q, \tag{17.7}$$

by homogeneity. This is *Hölder's inequality.*

The analogous statement also holds for $p = 1$ and $q = 1$, and was implicitly used in the previous two sections.

18 $\ell^p(E)^*$

Let $1 < p, q < \infty$ be conjugate exponents again, and let h be a q-summable function on a nonempty set E. If f is a p-summable function on E, then $f\,h$ is summable, and we can put

$$\lambda_h(f) = \sum_{x \in E} f(x)\,h(x). \tag{18.1}$$

This defines a bounded linear functional on $\ell^p(E)$, since

$$|\lambda_h(f)| \le \|h\|_q \, \|f\|_p, \tag{18.2}$$

by Hölder's inequality.

If we choose f so that

$$f(x)\, h(x) = |h(x)|^q \tag{18.3}$$

for every $x \in E$, and $f(x) = 0$ when $h(x) = 0$, then

$$|f(x)| = |h(x)|^{q-1}, \tag{18.4}$$

and

$$|f(x)|^p = |h(x)|^q, \tag{18.5}$$

since $p(q-1) = q$. In this case,

$$\lambda_h(f) = \|h\|_q^q = \|h\|_q \, \|f\|_p, \tag{18.6}$$

and it follows that the dual norm of λ_h on $\ell^p(E)$ is $\|h\|_q$.

Now let λ be any bounded linear functional on $\ell^p(E)$, and let h be the function on E defined by $h(y) = \lambda(\delta_y)$, so that

$$\lambda(f) = \sum_{x \in E} f(x)\, h(x) \tag{18.7}$$

when $f(x) \ne 0$ for only finitely many $x \in E$. If $E_1 \subseteq E$ has only finitely many elements, then we can choose f so that $f(x) = 0$ when $x \in E \backslash E_1$ or $h(x) = 0$ and (18.3) holds for every $x \in E_1$, and we get that

$$\lambda_h(f) = \sum_{x \in E_1} |h(x)|^q. \tag{18.8}$$

We also have that

$$\|f\|_p = \Big(\sum_{x \in E_1} |h(x)|^q \Big)^{1/p}, \tag{18.9}$$

which implies that

$$\Big(\sum_{x \in E_1} |h(x)|^q \Big)^{1/q} \tag{18.10}$$

is bounded by the dual norm of λ on $\ell^p(E)$. It follows that h is q-summable, and that $\|h\|_q$ is less than or equal to the dual norm of λ on $\ell^p(E)$.

The remaining point is that $\lambda(f) = \lambda_h(f)$ for every $f \in \ell^p(E)$. This holds by construction when $f(x) \ne 0$ for only finitely many $x \in E$, and it is easy to see that these functions are dense in $\ell^p(E)$. Thus $\lambda(f) = \lambda_h(f)$ for every $f \in \ell^p(E)$ by continuity.

19 Separability

Remember that a metric space is said to be *separable* if it contains a dense set with only finitely or countably many elements. For example, the real line is separable, because the rationals are countable and dense. Similarly, $\mathbf{R}^n$ and $\mathbf{C}^n$ are separable for each positive integer n.

Let V be a real or complex vector space equipped with a norm. If there is a set $A \subseteq V$ with only finitely or countably many elements whose span is dense in V, then V is separable. In the real case, the set of linear combinations of elements of A with rational coefficients is a countable dense set in V. In the complex case, one can use coefficients with rational real and imaginary parts.

For example, $\ell^p(E)$ is separable when E is countable and $1 \le p < \infty$, because the linear span of the functions δ_y, $y \in E$, is dense. The same argument applies to $c_0(E)$ equipped with the supremum norm, and to $\ell^p(E)$ with $0 < p < 1$, using the metric $\|f_1 - f_2\|_p^p$. The space of continuous functions on the unit interval is separable with respect to the supremum norm, because polynomials are dense.

One can show that the space of continuous functions on any compact metric space is separable with respect to the supremum norm. This uses the fact that continuous functions on compact metric spaces are automatically uniformly continuous.

20 An extension lemma

Let V be a real vector space, let W be a linear subspace of codimension 1 in V, and let z be an element of $V \backslash W$. Thus every $v \in V$ can be expressed in a unique way as

$$v = w + t\,z, \tag{20.1}$$

where $w \in W$ and $t \in \mathbf{R}$. If λ is a linear functional on W, then any extension of λ to a linear functional μ on V is uniquely determined by its value $\alpha \in \mathbf{R}$ at z, since

$$\mu(w + t\,z) = \lambda(w) + t\,\alpha. \tag{20.2}$$

Any choice of $\alpha \in \mathbf{R}$ defines an extension μ of λ in this way.

Suppose now that V is equipped with a norm $\|\cdot\|$, and that λ is a bounded linear functional with respect to the restriction of $\|\cdot\|$ to W. Thus there is an $L \ge 0$ such that

$$|\lambda(w)| \le L\,\|w\| \tag{20.3}$$

for every $w \in W$. We would like to choose $\alpha \in \mathbf{R}$ and hence μ so that

$$|\mu(v)| \le L\,\|v\| \tag{20.4}$$

for every $v \in V$, which is the same as

$$|\lambda(w) + t\,\alpha| \le L\,\|w + t\,z\| \tag{20.5}$$

for every $w \in W$ and $t \in \mathbf{R}$. This holds by hypothesis when $t = 0$, and for $t \neq 0$ it is enough to show that

$$|\lambda(w) + \alpha| \le L\,\|w + z\| \tag{20.6}$$

for every $w \in W$, since otherwise we can replace w by $t\,w$ and then take out $|t|$ from both sides by homogeneity.

Equivalently, we would like to choose $\alpha \in \mathbf{R}$ such that

$$-L\,\|w + z\| - \lambda(w) \le \alpha \le L\,\|w + z\| - \lambda(w) \tag{20.7}$$

for every $w \in W$. To do this, it suffices to show that

$$-L\,\|w_1 + z\| - \lambda(w_1) \le L\,\|w_2 + z\| - \lambda(w_2) \tag{20.8}$$

for every $w_1, w_2 \in W$. By hypothesis, we know that

$$\lambda(w_2 - w_1) \le L\,\|w_2 - w_1\| \tag{20.9}$$

for every $w_1, w_2 \in W$, which implies that

$$\lambda(w_2 - w_1) \le L\,\|w_2 + z\| + L\,\|w_1 + z\|, \tag{20.10}$$

because of the triangle inequality. This gives the desired estimate.

21 The Hahn–Banach theorem

Let V be a real vector space equipped with a norm $\|\cdot\|$, and let W be a linear subspace of V. The *Hahn–Banach theorem* asserts that every bounded linear functional λ on W has an extension to a bounded linear functional μ on V with the same norm.

If W has finite codimension in V, then one can apply the procedure described in the previous section finitely many times. If V is separable, then one can apply this procedure repeatedly to extend λ to a dense linear subspace of V, and then to all of V by continuity. Otherwise, there is an argument based on the axiom of choice.

As an application, it follows that for every $v \in V$ there is a bounded linear functional μ on V with norm 1 such that

$$\mu(v) = \|v\|. \tag{21.1}$$

To see this, one can start with the linear functional on the 1-dimensional subspace spanned by v defined by

$$\lambda(r\,v) = r\,\|v\|,\ \ r \in \mathbf{R}. \tag{21.2}$$

This has norm 1, and a norm-1 extension to V has the desired properties.

Similarly, if W_0 is a closed linear subspace of V and $v \in V \backslash W_0$, then there is a bounded linear functional μ on V such that

$$\mu(v) \neq 0 \tag{21.3}$$

and

$$\mu(w) = 0 \tag{21.4}$$

for every $w \in W_0$. For it is easy to find a bounded linear functional on the span of W_0 and v with these properties, which can then be extended to V.

22 Complex vector spaces

Let V be a complex vector space, which is then also a real vector space, since we can simply forget about multiplication by i. If λ is a linear functional on V as a complex vector space, then the real part of λ is a linear functional on V as a real vector space. The imaginary part of λ can be recovered from the real part, by the formula

$$\operatorname{Im} \lambda(v) = i \operatorname{Re} \lambda(-i\, v). \tag{22.1}$$

Similarly, any linear functional on V as a real vector space can be expressed as the real part of a linear functional on V as a complex vector space.

Suppose that V is equipped with a norm $\| \cdot \|$, which is also a norm on V as a real vector space. If λ is a bounded linear functional on V as a complex vector space, then one can show that the norm of λ is equal to the norm of the real part of λ as a bounded linear functional on V as a real vector space. This uses the fact that the norm of any $v \in V$ is equal to the norm of $c\, v$ for every complex number c with $|c| = 1$. Also, the modulus of a complex number z is the same as the maximum of the real parts of the complex numbers $c\, z$ with $c \in \mathbf{C}$ and $|c| = 1$. Every bounded linear functional on V as a real vector space is the real part of a bounded linear functional on V as a complex vector space, as in the previous paragraph.

This permits questions about linear functionals on complex vector spaces to be reduced to questions about linear functionals on real vector spaces. In particular, the Hahn–Banach theorem also applies to complex vector spaces.

23 The weak* topology

Let V be a real or complex vector space equipped with a norm $\|v\|$, and let V^* be the dual space of bounded linear functionals on V with the dual norm $\|\lambda\|_*$. As usual, the dual norm determines a metric on V^*, and hence a topology on V^*. We are also interested in another topology on V^*, known as the *weak* topology*.

A set $U \subseteq V^*$ is an open set in the weak* topology if for every $\lambda \in U$ there are finitely many vectors $v_1, \ldots, v_n \in V$ and positive real numbers $r_1, \ldots, r_n$ such that

$$\{\mu \in V^* : |\mu(v_j) - \lambda(v_j)| < r_j,\ 1 \leq j \leq n\} \subseteq U. \tag{23.1}$$

Equivalently,

$$\{\mu \in V^* : |\mu(v_j) - \lambda(v_j)| < r_j,\ 1 \le j \le n\} \tag{23.2}$$

is an open set in V^* with respect to the weak* topology for every $\lambda \in V^*$, $v_1, \ldots, v_n \in V$, and $r_1, \ldots, r_n > 0$, and these sets form a base for the weak*-topology.

If $U \subseteq V^*$ is an open set with respect to the weak* topology, then U is also an open set with respect to the topology determined by the dual norm. However, open balls in V^* with respect to the dual norm are not open sets in the weak* topology, unless V is finite-dimensional.

For each $v \in V$,

$$\lambda \mapsto \lambda(v) \tag{23.3}$$

defines a linear functional on V^*. More precisely, this is a bounded linear functional on V^* with respect to the dual norm, since

$$|\lambda(v)| \le \|\lambda\|_* \, \|v\|, \tag{23.4}$$

by definition of the dual norm. The norm of this linear functional on V^* is actually equal to $\|v\|$, by the Hahn–Banach theorem. These linear functionals are also continuous on V^* with respect to the weak* topology, by definition of the weak* topology. The weak* topology can be described as the weakest topology on V^* in which these linear functionals are continuous.

24 Seminorms

A nonnegative real-valued function $N(w)$ on a real or complex vector space W is said to be a *seminorm* if

$$N(t\, w) = |t| \, N(w) \tag{24.1}$$

for every $w \in W$ and $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, and

$$N(w + z) \le N(w) + N(z) \tag{24.2}$$

for every $w, z \in W$. These are the same conditions as for a norm, except that $N(w) = 0$ may occur when $w \ne 0$.

Suppose that $\mathcal{N}$ is a collection of seminorms on W which satisfies the positivity condition that for every $w \in W$ with $w \ne 0$,

$$N(w) > 0 \tag{24.3}$$

for some $N \in \mathcal{N}$. As in the case of a single norm, there is a nice topology on W determined by $\mathcal{N}$, Specifically, $U \subseteq W$ is an open set in this topology if for every $u \in U$ there are finitely many seminorms $N_1, \ldots, N_l \in \mathcal{N}$ and positive real numbers $r_1, \ldots, r_l$ such that

$$\{w \in W : N_j(w - u) < r_j,\ 1 \le j \le l\} \subseteq U. \tag{24.4}$$

Equivalently,

$$\{w \in W : N_j(w - u) < r_j,\ 1 \le j \le l\} \tag{24.5}$$

is an open set in W with respect to this topology for every $u \in W$, $N_1, \ldots, N_l$ in $\mathcal{N}$, and $r_1, \ldots, r_l > 0$, and these sets form a base for the topology. The positivity condition ensures that this topology is Hausdorff.

For example, if μ is a linear functional on W, then

$$N(w) = |\mu(w)| \tag{24.6}$$

is a seminorm on W. The weak* topology on the dual V^* of a vector space V corresponds to the family of seminorms

$$N_v(\lambda) = |\lambda(v)|,\ v \in V. \tag{24.7}$$

If $\mathcal{N}$ consists of only finitely many seminorms $N_1, \ldots, N_k$ on W, then their maximum

$$N(w) = \max(N_1(w), \ldots, N_k(w)) \tag{24.8}$$

is a norm on W, and the topology on W corresponding to $\mathcal{N}$ is the same as the usual one determined by N.

If $\mathcal{N}$ consists of an infinite sequence $N_1, N_2, \ldots$ of seminorms on W, then the corresponding topology on W can be described by a metric. For each $j \ge 1$, one can check that

$$\rho_j(u, w) = \min(N_j(u - w), 1/j) \tag{24.9}$$

is a semimetric on W. This means that $\rho_j(u, w)$ satisfies the same conditions as a metric, except that $\rho_j(u, w) = 0$ can occur when $u \ne w$. One can also check that

$$\rho(u, w) = \max\{\rho_j(u, w) : j \ge 1\} \tag{24.10}$$

is a metric on W, and that the topology on W determined by this metric is the same as the topology corresponding to the sequence of seminorms as before.

25 The weak* topology, 2

Let V be a real or complex vector space with a norm $\|v\|$, and let V^* be the dual space of bounded linear functionals on V with the dual norm $\|\lambda\|_*$. Also let

$$B^* = \{\lambda \in V^* : \|\lambda\|_* \le 1\} \tag{25.1}$$

be the closed unit ball in V^*.

As in the previous section, the weak* topology on V^* is the same as the topology corresponding to the family of seminorms

$$N_v(\lambda) = |\lambda(v)|,\ v \in V. \tag{25.2}$$

For $A \subseteq V$, consider the family of seminorms $\mathcal{N}_A$ on V^* consisting of N_v with $v \in A$. If the linear span of A is dense in V, then the topology on B^* induced by the weak* topology on V^* is the same as the topology induced by the one corresponding to $\mathcal{N}_A$. This is easy to see, and it works just as well for any subset of V^* which is bounded with respect to the dual norm.

If V is separable, then we can take A to be a countable set. As in the previous section, the topology corresponding to $\mathcal{N}_A$ is then determined by a metric. Thus the topology on B^* or other bounded subsets of V^* induced by the weak* topology can be described by a metric in this case.

Technically, the term "weak* topology" on V^* should perhaps only be used when V is complete. If V is not complete, then V can be isometrically embedded as a dense linear subspace of a Banach space. Although V and its completion have the same dual space, the additional elements of the completion can affect the weak* topology on the dual. However, this does not matter for the topology on bounded subsets of the dual induced by the weak* topology, by the earlier remarks.

26 The Banach–Alaoglu theorem

Let V be a real or complex vector space equipped with a norm $\|v\|$, and let V^* be the dual space with dual norm $\|\lambda\|_*$. The *Banach–Alaoglu theorem* states that the closed unit ball B^* in V^* is compact with respect to the weak* topology.

As usual, each $v \in V$ determines a linear functional

$$\lambda \mapsto \lambda(v) \tag{26.1}$$

on V^*. This defines a mapping from V^* into the Cartesian product of copies of $\mathbf{R}$ or $\mathbf{C}$, as appropriate, parameterized by $v \in V$. One can check that this mapping is a homeomorphism from V^* onto its image in the Cartesian product with respect to the topology induced by the product topology, because of the way that the weak* topology is defined. One can also check that the image of B^* under this mapping is a closed set with respect to the product topology. The image of B^* is also contained in the product of closed intervals or disks, as appropriate, which implies compactness.

If V is separable, then one can argue more directly that every sequence $\{\lambda_j\}_{j=1}^\infty$ in B^* has a subsequence that converges to an element of B^* in the weak* topology. For each $v \in V$, one can use the compactness of closed and bounded subsets of $\mathbf{R}$ or $\mathbf{C}$ to get a subsequence $\{\lambda_{j_l}\}_{l=1}^\infty$ of $\{\lambda_j\}_{j=1}^\infty$ such that $\{\lambda_{j_l}(v)\}_{j=1}^\infty$ converges. By standard arguments, this can be refined to show that there is a subsequence which converges pointwise on a countable subset of V. Hence there is a subsequence that converges on a dense set in V when V is separable. Using the fact that these linear functionals all have norm less than or equal to 1, one can show that this subsequence actually converges pointwise everywhere on V, and that the limit is also a linear functional with norm less than or equal to 1.

27 The weak topology

Let V be a real or complex vector space equipped with a norm $\|v\|$. As in Section 24 each bounded linear functional λ on V determines a seminorm on V, given by

$$N_\lambda(v) = |\lambda(v)|. \tag{27.1}$$

The topology on V defined by the family of seminorms N_λ, $\lambda \in V^*$, is called the *weak topology*. Note that for each $v \in V$ there is a $\lambda \in V^*$ such that $\lambda(v) \neq 0$ and hence $N_\lambda(v) > 0$, by the Hahn–Banach theorem, so that this family of seminorms satisfies the usual positivity condition.

If $U \subseteq V$ is an open set with respect to the weak topology, then U is also an open set with respect to the topology associated to the norm. However, an open ball in V is not an open set with respect to the weak topology, unless V has finite dimension. If W is a linear subspace of V which is closed in the norm topology, then W is also closed in the weak topology, by the Hahn–Banach theorem. Each bounded linear functional on V is also continuous with respect to the weak topology, and the weak topology is the weakest topology on V with this property.

The weak topology can also be defined on the dual V^* of V, using bounded linear functionals on V^* with respect to the dual norm. As in Section 23,

$$\lambda \mapsto \lambda(v) \tag{27.2}$$

defines a bounded linear functional on V^* for each $v \in V$, which implies that every open set in V^* with respect to the weak* topology is also an open set in the weak topology on V^*. If every bounded linear functional on V^* is given by evaluation at an element of V in this way, then V is said to be *reflexive*. Thus the weak and weak* topologies on V^* are the same when V is reflexive.

Hilbert spaces are automatically reflexive, because of the way that they can be identified with their dual spaces. If E is a nonempty set and $1 < p < \infty$, then $\ell^p(E)$ is also reflexive. As in Section 18, the dual of $\ell^p(E)$ can be identified with $\ell^q(E)$, where q is the conjugate exponent corresponding to p. For the same reasons, the dual of $\ell^q(E)$ can be identified with $\ell^p(E)$. As in Sections 15 and 16, the dual of $c_0(E)$ can be identified with $\ell^1(E)$ and the dual of $\ell^1(E)$ can be identified with $\ell^\infty(E)$, which is not the same as $c_0(E)$ when E has infinitely many elements.

28 Seminorms, 2

Let W be a real or complex vector space, and let $\mathcal{N}$ be a family of seminorms on W that satisfies the positivity condition that for each $w \in W$ there is an $N \in \mathcal{N}$ such that

$$N(w) > 0. \tag{28.1}$$

As in Section 24, this leads to a nice topology on W. Let us say that a linear functional ψ on W is *bounded* with respect to $\mathcal{N}$ if there are finitely many seminorms $N_1, \ldots, N_l \in \mathcal{N}$ and $A \ge 0$ such that

$$|\psi(w)| \le A \max(N_1(w), \ldots, N_l(w)) \tag{28.2}$$

for each $w \in W$. This implies that

$$|\psi(u) - \psi(w)| \le A \max(N_1(u - w), \ldots, N_l(u - w)) \tag{28.3}$$

for every $u, w \in W$, and it is easy to see that ψ is therefore continuous with respect to the topology on W corresponding to $\mathcal{N}$.

Conversely, suppose that ψ is a linear functional on W which is continuous with respect to the topology corresponding to $\mathcal{N}$. This implies that

$$U = \{w \in w : |\psi(w)| < 1\} \tag{28.4}$$

is an open set in this topology. Since $0 \in U$, this means that there are finitely many seminorms $N_1, \ldots, N_l \in \mathcal{N}$ and positive real numbers $r_1, \ldots, r_l > 0$ such that

$$\{w \in w : N_j(w) < r_j,\ 1 \le j \le l\} \subseteq U, \tag{28.5}$$

by definition of the topology on W corresponding to $\mathcal{N}$. Equivalently, $|\psi(w)| < 1$ when

$$N_1(w) < r_1, \ldots, N_l(w) < r_l, \tag{28.6}$$

which implies that

$$|\psi(w)| \le \max(r_1^{-1}\, N_1(w), \ldots, r_l^{-1}\, N_l(w)), \tag{28.7}$$

and hence that ψ is bounded with respect to $\mathcal{N}$.

For example, let Φ be a collection of linear functionals on W with the property that for each $w \in W$ there is a $\phi \in \Phi$ such that $\phi(w) \ne 0$. Let $\mathcal{N}_\Phi$ be the family of seminorms

$$N_\phi(w) = |\phi(w)| \tag{28.8}$$

on W, $\phi \in \Phi$, which satisfies the positivity condition by hypothesis. Each $\phi \in \Phi$ is a continuous linear functional on W with respect to the topology corresponding to $\mathcal{N}_\Phi$, as well as any finite linear combination of elements of Φ.

Conversely, any linear functional ψ on W which is continuous with respect to the topology associated to $\mathcal{N}_\Phi$ can be expressed as a linear combination of finitely many elements of Φ. For if ψ is continuous, then it is bounded with respect to $\mathcal{N}_\Phi$, and there are $\phi_1, \ldots, \phi_l \in \Phi$ and $A \ge 0$ such that

$$|\psi(w)| \le A \max(|\phi_1(w)|, \ldots, |\phi_l(w)|) \tag{28.9}$$

for each $w \in W$. In particular, $\psi(w) = 0$ when

$$\phi_1(w) = \cdots = \phi_l(w) = 0. \tag{28.10}$$

It follows from standard arguments in linear algebra that ψ is a linear combination of $\phi_1, \ldots, \phi_l$, as desired.

29 Convergent sequences

Let $(M, d(\cdot,\cdot))$ be a metric space. By definition, a sequence $\{x_j\}_{j=1}^\infty$ of elements of M converges to $x \in M$ if and only if

$$\lim_{j\to\infty} d(x_j, x) = 0 \tag{29.1}$$

as a sequence of real numbers. In particular, if W is a vector space with a norm $\|\cdot\|_W$, then a sequence of vectors $\{w_j\}_{j=1}^\infty$ converges to $w \in W$ with respect to the norm if and only if

$$\lim_{j\to\infty} \|w_j - w\|_W = 0. \tag{29.2}$$

If instead the topology on W is determined by a family $\mathcal{N}$ of seminorms on W, then one can check that $\{w_j\}_{j=1}^\infty$ converges to w if and only if

$$\lim_{j\to\infty} N(w_j - w) = 0 \tag{29.3}$$

for each $N \in \mathcal{N}$.

Let V be a real or complex vector space equipped with a norm $\|\cdot\|$, and let V^* be the dual space of bounded linear functionals on V equipped with the dual norm $\|\cdot\|_*$. A sequence $\{v_j\}_{j=1}^\infty$ of vectors in V converges to $v \in V$ in the weak topology if and only if

$$\lim_{j\to\infty} \lambda(v_j) = \lambda(v) \tag{29.4}$$

as a sequence of real or complex numbers for each $\lambda \in V^*$. Similarly, a sequence $\{\lambda_j\}_{j=1}^\infty$ of bounded linear functionals on V converges to $\lambda \in V^*$ in the weak* topology if and only if

$$\lim_{j\to\infty} \lambda_j(v) = \lambda(v) \tag{29.5}$$

as a sequence of real or complex numbers for each $v \in V$.

In a metric space M, the topology is determined by convergence of sequences, because closed subsets of M can be characterized in terms of convergent sequences. This also works in topological spaces which satisfy the first countability condition, for which there is a countable local base for the topology at every point, but it does not work in arbitrary topological spaces. In some circumstances, the induced topology on interesting subsets of a topological space X may satisfy the first countability condition, even if X does not enjoy this property itself. For example, this is the case for bounded subsets of the dual V^* of a separable vector space V with a norm, with respect to the weak* topology on V^*.

Suppose that $\{\lambda_j\}_{j=1}^\infty$ is a sequence of bounded linear functionals on a vector space V with a norm which is uniformly bounded in the sense that

$$\|\lambda_j\|_* \le L \tag{29.6}$$

for some $L \ge 0$ and each $j \ge 1$. If $\{\lambda_j(v)\}_{j=1}^\infty$ converges as a sequence of real or complex numbers for a set of $v \in V$ whose linear span is dense in V, then one can show that $\{\lambda_j(v)\}_{j=1}^\infty$ is a Cauchy sequence in $\mathbf{R}$ or $\mathbf{C}$ for every $v \in V$, and hence converges. If we put

$$\lambda(v) = \lim_{j \to \infty} \lambda_j(v) \tag{29.7}$$

for each $v \in V$, then it is easy to see that λ is a bounded linear functional on V, with

$$\|\lambda\|_* \le L, \tag{29.8}$$

and $\{\lambda_j\}_{j=1}^\infty$ converges to λ in the weak* topology.

30 Uniform boundedness

Let V be a real or complex vector space with a norm $\|\cdot\|$, and let V^* be the dual space of bounded linear functionals on V with the dual norm $\|\cdot\|_*$. Suppose that $\{\lambda_j\}_{j=1}^\infty$ is a sequence in V^* such that $\{\lambda_j(v)\}_{j=1}^\infty$ converges in $\mathbf{R}$ or $\mathbf{C}$ for every $v \in V$, which implies that $\{\lambda_j(v)\}_{j=1}^\infty$ is bounded for each $v \in V$. If V is complete, then a theorem of Banach and Steinhaus implies that $\{\lambda_j\}_{j=1}^\infty$ is bounded in V^*, so that

$$\|\lambda_j\|_* \le L \tag{30.1}$$

for some $L \ge 0$ and every $j \ge 1$.

To see this, put

$$A(n) = \{v \in V : |\lambda_j(v)| \le n \text{ for every } j \ge 1\} \tag{30.2}$$

for each positive integer n. Because λ_j is bounded and hence continuous for every j, $A(n)$ is a closed set in V for each n, while

$$\bigcup_{n=1}^\infty A(n) = V, \tag{30.3}$$

by hypothesis. Hence $A(n)$ contains a nonempty open set in V for some n, by the Baire category theorem, since V is complete. Thus the λ_j's are uniformly bounded on a nonempty open set in V, and one can use this and linearity to show that their dual norms are bounded.

Similarly, if $\{v_j\}_{j=1}^\infty$ is a sequence in V such that $\{\lambda(v_j)\}_{j=1}^\infty$ converges in $\mathbf{R}$ or $\mathbf{C}$ for every $\lambda \in V^*$, then one can show that the v_j's have bounded norm. As in the previous case, it is actually sufficient to know that $\{\lambda(v_j)\}_{j=1}^\infty$ is bounded for each $\lambda \in V^*$. More precisely, one can first use the Baire category theorem to show that there is an $L' \ge 0$ such that

$$|\lambda(v_j)| \le L' \, \|\lambda\|_* \tag{30.4}$$

for every $\lambda \in V^*$ and $j \ge 1$, as before, and then the Hahn–Banach theorem implies that

$$\|v_j\| \le L' \tag{30.5}$$

for each $j \ge 1$, as desired.

31 Examples

Let E be a nonempty set, and let $1 \le p, q \le \infty$ be conjugate exponents. As in Sections 15, 16, and 18, the dual of $\ell^p(E)$ can be identified with $\ell^q(E)$ when $1 \le p < \infty$, and the dual of $c_0(E)$ with the ℓ^∞ norm can be identified with $\ell^1(E)$. Hence the weak* topology can be defined on $\ell^q(E)$ for $1 \le q \le \infty$ using these identifications.

By definition, a sequence $\{f_j\}_{j=1}^\infty$ of functions on E in $\ell^q(E)$ converges to another function $f \in \ell^q(E)$ in the weak* topology if

$$\lim_{j \to \infty} \sum_{x \in E} f_j(x)\, h(x) = \sum_{x \in E} f(x)\, h(x) \tag{31.1}$$

for every $h \in \ell^p(E)$ when $1 \le p < \infty$, and every $h \in c_0(E)$ when $p = \infty$. This implies that

$$\lim_{j \to \infty} f_j(x) = f(x) \tag{31.2}$$

for every $x \in E$, by taking h to be equal to 1 at x and 0 elsewhere. Moreover, weak* convergence implies that there is an $L \ge 0$ such that

$$\|f_j\|_q \le L, \tag{31.3}$$

as in the previous section.

Conversely, (31.2) and (31.3) imply that $f \in \ell^q(E)$, with

$$\|f\|_q \le L. \tag{31.4}$$

This is a straightforward consequence of the definitions when $q = \infty$, while for $1 \le q < \infty$ one can observe first that

$$\Big(\sum_{x \in E_1} |f(x)|^q\Big)^{1/q} = \lim_{j \to \infty} \Big(\sum_{x \in E_1} |f_j(x)|^q\Big)^{1/q} \le L \tag{31.5}$$

for every set $E_1 \subseteq E$ with only finitely many elements. Of course, (31.2) implies (31.1) when $h(x) \ne 0$ for only finitely many $x \in E$. One can extend this to all $h \in \ell^p(E)$ when $1 \le p < \infty$ and to all $h \in c_0(E)$ when $p = \infty$ by approximation, using the uniform bound (31.3).

Similarly, a sequence $\{f_j\}_{j=1}^\infty$ of summable functions on E converges to a summable function f on E in the weak topology on $\ell^1(E)$ if (31.1) holds for every $h \in \ell^\infty(E)$. In particular, this implies that

$$\lim_{j \to \infty} \sum_{x \in E} f_j(x) = \sum_{x \in E} f(x). \tag{31.6}$$

For instance, if $\{f_j\}_{j=1}^\infty$ is a sequence of nonnegative real-valued functions on E which converges weakly to 0, then (31.6) implies that

$$\lim_{j \to \infty} \|f_j\|_1 = 0. \tag{31.7}$$

However, it is easy to give examples of sequences of nonnegative real-valued summable functions that converge to 0 in the weak* topology and not in the ℓ^1 norm.

32 An embedding

Let V be a vector space with norm $\|v\|$, and let V^* be its dual space with dual norm $\|\lambda\|_*$. Also let X be the closed unit ball in V^*, consisting of $\lambda \in V^*$ with $\|\lambda\|_* \le 1$, equipped with the weak* topology. Thus X is a compact Hausdorff space, by the Banach–Alaoglu theorem.

As usual, each $v \in V$ determines a bounded linear functional η_v on V^*, defined by

$$\eta_v(\lambda) = \lambda(v). \tag{32.1}$$

The restriction of η_v to the unit ball of V^* defines a continuous function on X for each $v \in V$, so that we get a linear mapping from V into the space $\mathcal{C}(X)$ of continuous real or complex-valued functions on X, as appropriate. This mapping is an isometry, since for each $v \in V$,

$$\|\eta_v\|_\infty = \|\eta_v\|_* = \|v\|. \tag{32.2}$$

Here $\|\eta\|_\infty$ denotes the supremum norm of a continuous function η on X.

If V is separable, then the topology on X can be described by a metric, as in Section 25. In particular, X is separable, since it is compact. If E is a countable dense set in X, then there is a natural isometric embedding of $\mathcal{C}(X)$ into $\ell^\infty(E)$, defined by restricting continuous functions on X to E. Hence we get an isometric embedding of $\mathcal{C}(X)$ into $\ell^\infty(E)$ too.

33 Induced mappings

Let X, Y be topological spaces, and let $\phi : X \to Y$ be a continuous mapping between them. If f is a continuous function on Y, then

$$\Phi(f) = f \circ \phi \tag{33.1}$$

is a continuous function on X, and Φ determines a linear mapping from $\mathcal{C}_b(Y)$ into $\mathcal{C}_b(X)$. Moreover,

$$\|\Phi(f)\|_{X,\infty} \le \|f\|_{Y,\infty}, \tag{33.2}$$

where the subscripts indicate on which space the supremum norm is defined, so that $\Phi : \mathcal{C}_b(Y) \to \mathcal{C}_b(X)$ is a bounded linear operator with norm ≤ 1. It is easy to see that Φ actually has operator norm equal to 1, since Φ takes constant functions on Y to constant functions on X.

If $\phi(X)$ is dense in Y, then

$$\|\Phi(f)\|_{X,\infty} = \|f\|_{Y,\infty} \tag{33.3}$$

for each bounded continuous function f on Y. Thus Φ is an isometric embedding of $\mathcal{C}_b(Y)$ into $\mathcal{C}_b(X)$ in this case. For example, one can apply this to the coordinate projections from the unit square $[0,1] \times [0,1]$ onto the unit interval $[0,1]$ to get isometric embeddings of $\mathcal{C}([0,1])$ into $\mathcal{C}([0,1] \times [0,1])$. One can also apply this to a continuous mapping ϕ from the unit interval $[0,1]$ onto the unit square $[0,1] \times [0,1]$, which is to say a space-filling curve, to get an isometric embedding of $\mathcal{C}([0,1] \times [0,1])$ into $\mathcal{C}([0,1])$.

There are sometimes linear operators which send continuous functions on a closed set $A \subseteq X$ to continuous extensions on X, which can lead to embeddings of $\mathcal{C}_b(A)$ into $\mathcal{C}_b(X)$. For example, if X is the unit interval $[0,1]$, then a continuous function f on a closed set $A \subseteq [0,1]$ can be extended to a continuous function on $[0,1]$ which is linear on the intervals in $[0,1]\backslash A$. If A does not contain 0 or 1, then the extension can be taken to be constant on the corresponding interval in the complement. This leads to an isometric linear embedding of $\mathcal{C}(A)$ into $\mathcal{C}(0,1)$.

In particular, this applies to the middle-thirds Cantor set. If X is a compact metric space, then there is a continuous mapping from the Cantor set onto X. It follows that there is an isometric linear embedding of $\mathcal{C}(X)$ into $\mathcal{C}(A)$, and hence into $\mathcal{C}([0,1])$. Using also the embedding described in the previous section, any separable Banach space can be isometrically embedded into $\mathcal{C}([0,1])$.

34 Continuity of norms

Let V be a real or complex vector space equipped with a norm $\|v\|$. By the triangle inequality,

$$\|v\| \le \|w\| + \|v - w\| \tag{34.1}$$

and

$$\|w\| \le \|v\| + \|v - w\| \tag{34.2}$$

for each $v, w \in V$, and therefore

$$\big|\|v\| - \|w\|\big| \le \|v - w\| \tag{34.3}$$

for every $v, w \in V$. This implies that $\|v\|$ is a continuous function on V, with respect to the topology determined by the norm. Similarly, if $\mathcal{N}$ is a family of seminorms on V, then each seminorm $N \in \mathcal{N}$ is continuous with respect to the topology determined by $\mathcal{N}$.

It is easy to show that any norm on $\mathbf{R}^n$ or $\mathbf{C}^n$ is bounded by a constant multiple of the standard norm, by expressing vectors as linear combinations of the standard basis and applying the triangle inequality. Hence the norm is continuous with respect to the standard topology on $\mathbf{R}^n$ or $\mathbf{C}^n$. Because the unit sphere with respect to the standard norm is compact, any continuous function on the sphere attains its minimum, which is positive in the case of another norm. It follows that any norm on $\mathbf{R}^n$ or $\mathbf{C}^n$ is also bounded from below by a positive multiple of the standard norm, so that they are equivalent. If V is a finite-dimensional real or complex vector space, then V is isomorphic to $\mathbf{R}^n$ or $\mathbf{C}^n$ for some n, and the preceding remarks imply that any two norms on V are equivalent.

If V is an infinite-dimensional vector space equipped with a norm $\|v\|$, then $\|v\|$ is not a continuous function with respect to the weak topology, because open balls are not open subsets of V in the weak topology. However, closed balls are closed sets in the weak topology, which means that $\|v\|$ is lower semicontinuous with respect to the weak topology. Similarly, closed balls in the dual space V^* relative to the dual norm $\|\lambda\|_*$ are closed sets with respect to the weak* topology, so that the dual norm is lower semicontinuous with respect to the weak* topology.

Every linear functional on $\mathbf{R}^n$ or $\mathbf{C}^n$ is bounded relative to the standard norm, as one can see using the standard basis again. Thus every linear functional on a finite-dimensional vector space V is bounded relative to any norm on V, by the earlier remarks, and the weak topology on V is the same as the topology determined by the norm. In this case, the dual space V^* has the same finite dimension as V, and the weak* topology on V^* is the same as the topology determined by the dual norm. Of course, finite-dimensional spaces are automatically reflexive.

35 Infinite series

Let V be a real or complex vector space equipped with a norm $\|v\|_V$. An infinite series

$$\sum_{j=1}^{\infty} v_j \tag{35.1}$$

with terms in V is said to *converge* if the corresponding sequence of partial sums

$$\sum_{j=1}^{n} v_j \tag{35.2}$$

converges in V. In this case, the value of the sum (35.1) is defined to be the limit of the sequence of partial sums.

Similarly, (35.1) is said to converge *absolutely* if

$$\sum_{j=1}^{\infty} \|v_j\| \tag{35.3}$$

converges in the usual sense as an infinite series of nonnegative real numbers. As in the case of infinite series of real or complex numbers, absolute convergence of an infinite series in V implies that the corresponding sequence of partial sums is a Cauchy sequence. If V is complete, then absolute convergence of a series in V implies that the series converges in V.

Conversely, if every absolutely convergent series in V converges, then V is complete. For let $\{w_j\}_{j=1}^{\infty}$ be a Cauchy sequence in V, which we would like to show converges. By the definition of a Cauchy sequence, there is a subsequence $\{w_{j_l}\}_{l=1}^{\infty}$ such that

$$\|w_{j_{l+1}} - w_{j_l}\| \le 2^{-l} \tag{35.4}$$

for each $l \ge 1$. Thus

$$\sum_{l=1}^{\infty} (w_{j_{l+1}} - w_{j_l}) \tag{35.5}$$

converges absolutely, and so it also converges in V by hypothesis. Of course,

$$\sum_{l=1}^{n}(w_{j_{l+1}} - w_{j_l}) = w_n - w_1 \tag{35.6}$$

for each n, so that convergence of the series implies that

$$\lim_{n\to\infty} w_{j_n} = w_1 + \sum_{l=1}^{\infty}(w_{j_{l+1}} - w_{j_l}). \tag{35.7}$$

In any metric space, a Cauchy sequence converges as soon as any of its subsequences converge, and to the same limit. It follows that $\{w_j\}_{j=1}^{\infty}$ also converges in V, as desired.

Convergence of infinite series in a vector space can also be defined when the topology on the vector space is determined by a family of seminorms, in terms of the convergence of the corresponding sequence of partial sums. In particular, this applies to convergence with respect to the weak and weak* topologies.

36 Some examples

Let us consider vector spaces of real or complex-valued functions on the set $\mathbf{Z}_+$ of positive integers. For each $j \ge 1$, let δ_j be the function on $\mathbf{Z}_+$ equal to 1 at j and 0 elsewhere, as usual. If f is any function on $\mathbf{Z}_+$, then

$$\sum_{j=1}^{\infty} f(j)\,\delta_j \tag{36.1}$$

converges to f pointwise on $\mathbf{Z}_+$.

If $f \in \ell^p(\mathbf{Z}_+)$, $1 \le p < \infty$, then (36.1) converges to f in the ℓ^p norm. If $f \in c_0(\mathbf{Z}_+)$, then (36.1) converges to f in the ℓ^∞. Conversely, if $f \in \ell^\infty(\mathbf{Z}_+)$ and (36.1) converges to f in the ℓ^∞ norm, then $f \in c_0(\mathbf{Z}_+)$. For any $f \in \ell^\infty(\mathbf{Z}_+)$, (36.1) converges to f in the weak* topology, where $\ell^\infty(\mathbf{Z}_+)$ is identified with the dual of $\ell^1(\mathbf{Z}_+)$.

If $f \in \ell^1(\mathbf{Z}_+)$, then (36.1) converges absolutely in $\ell^1(\mathbf{E})$. This does not work when $p > 1$, but there is a version of absolute convergence in the weak topology when $1 < p < \infty$, and in the weak* topology when $p = \infty$. More precisely, let $1 \le p, q \le \infty$ be conjugate exponents, and put

$$\lambda_h(f) = \sum_{x \in \mathbf{Z}_+} f(x)\,h(x) \tag{36.2}$$

when $f \in \ell^p(\mathbf{Z}_+)$ and $h \in \ell^q(\mathbf{Z}_+)$, which makes sense by Hölder's inequality. Thus

$$\sum_{j=1}^{\infty} |\lambda_h(f(j)\,\delta_j)| = \sum_{j=1}^{\infty} |f(j)|\,|h(j)| \le \|f\|_p\,\|h\|_q, \tag{36.3}$$

and in particular the series on the left converges absolutely when $f \in \ell^p(\mathbf{Z}_+)$ and $h \in \ell^q(\mathbf{Z}_+)$.

If $f \in \ell^p(\mathbf{Z}_+)$ and $0 < p < 1$, then the partial sums of (36.1) converge to f with respect to the usual metric $\|f_1 - f_2\|_p^p$. The appropriate version of absolute convergence for a series of functions $\sum_{j=1}^\infty a_j$ in $\ell^p(E)$ on any set E when $0 < p < 1$ is that

$$\sum_{j=1}^{\infty} \|a_j\|_p^p \tag{36.4}$$

converges as a series of nonnegative real numbers. This is satisfied by (36.1) when $f \in \ell^p(\mathbf{Z}_+)$, as in the $p = 1$ case.

37 Quotient spaces

Let V be a real or complex vector space, and let W be a linear subspace of V. The quotient space V/W is defined by identifying $v, v' \in V$ when

$$v - v' \in W. \tag{37.1}$$

More precisely, this defines an equivalence relation on V, and the elements of V/W represent equivalence classes associated to this equivalence relation. The operations of addition and scalar multiplication on V lead to similar operations on V/W, so that V/W is also a real or complex vector space, as appropriate. By construction, there is a quotient map

$$q : V \to V/W, \tag{37.2}$$

that assigns to each $v \in V$ the element $q(v)$ of V/W corresponding to the equivalence class in V containing v, and which is a linear mapping from V onto V/W with kernel W.

Let $\|v\|_V$ be a norm on V. For each $z \in V/W$, put

$$\|z\|_{V/W} = \inf\{\|v\|_V : v \in V,\, q(v) = z\}. \tag{37.3}$$

It is easy to check that this defines a seminorm on V/W which is a norm exactly when W is a closed linear subspace of V with respect to the topology determined by $\|\cdot\|_V$.

If V is complete with respect to $\|\cdot\|_V$ and W is a closed subspace of V, then V/W is complete with respect to the quotient norm. To see this, it suffices to show that any absolutely convergent series

$$\sum_{j=1}^{\infty} z_j \tag{37.4}$$

of elements of V/W converges in V/W, as in Section 35. For each $j \geq 1$, let v_j be an element of V such that $q(v_j) = z_j$ and

$$\|v_j\|_V < \|z_j\|_{V/W} + 2^{-j}. \tag{37.5}$$

Thus

$$\sum_{j=1}^{\infty} v_j \tag{37.6}$$

converges absolutely in V, and hence converges in V by completeness. This implies that (37.4) converges in V/W, with

$$\sum_{j=1}^{\infty} z_j = q\Big(\sum_{j=1}^{\infty} v_j\Big). \tag{37.7}$$

If V has finite dimension $\dim V$, then W and V/W have finite dimensions as well, and

$$\dim V = \dim W + \dim V/W. \tag{37.8}$$

For if $v_1, \ldots, v_n$ is a basis for V such that $v_1, \ldots, v_l$ is a basis for W, where

$$l = \dim W \le \dim V = n, \tag{37.9}$$

then one can check that $q(v_{l+1}), \ldots, q(v_n)$ is a basis for V/W. As an extension of this to infinite dimensions, one can say that if V is separable, then V/W is separable, because q maps a countable dense set in V to a countable dense set in V/W.

38 Quotients and duality

Let V be a real or complex vector space equipped with a norm $\|\cdot\|_V$, let W be a closed linear subspace of V, and let V/W be the corresponding quotient space with quotient norm $\|\cdot\|_{V/W}$. Consider also the linear subspace of the dual space V^* defined by

$$W^\perp = \{\lambda \in V^* : \lambda(w) = 0\}. \tag{38.1}$$

Note that $W^\perp$ is closed with respect to the topology on V^* determined by the dual norm, and even with respect to the weak* topology.

If μ is a bounded linear functional on V/W, then

$$\lambda = \mu \circ q \tag{38.2}$$

is a bounded linear functional on V. By construction, $\lambda \in W^\perp$, and one can check that every $\lambda \in W^\perp$ corresponds to a $\mu \in (V/W)^\perp$ in this way. One can also check that the dual norm of λ on V is equal to the dual norm of μ on V/W. This leads to a natural isometric equivalence

$$W^\perp \cong (V/W)^*. \tag{38.3}$$

Similarly, if ϕ is a bounded linear functional on V, then the restriction of ϕ to W defines a bounded linear functional ψ on W whose dual norm is less than or equal to the dual norm of ϕ. Every bounded linear functional ψ on W is the restriction of a bounded linear functional ϕ on V to W with the same norm, by the Hahn–Banach theorem. Of course, the restriction of ϕ to W is 0 exactly when $\phi \in W^\perp$. This leads to another natural isometric equivalence

$$W^* \cong V^*/W^\perp. \tag{38.4}$$

39 Dual mappings

Let V, W be vector spaces, both real or both complex, and equipped with norms $\|v\|_V$, $\|w\|_W$. If $T : V \to W$ is a bounded linear mapping, then the operator norm of T is equal to

$$\sup\{|\lambda(T(v))| : v \in V,\, \lambda \in W^*,\, \|v\|_V \le 1,\, \|\lambda\|_{W^*} \le 1\}. \tag{39.1}$$

Here $\|\lambda\|_{W^*}$ denotes the dual norm of the bounded linear functional λ on W. It is clear from the definitions that (39.1) is less than or equal to the operator norm of T, and equality follows from the Hahn–Banach theorem.

The dual mapping T^* sends a bounded linear functional λ on W to the bounded linear functional on V defined by

$$T^*(\lambda) = \lambda \circ T. \tag{39.2}$$

It is easy to see that $T^* : W^* \to V^*$ is a bounded linear operator with respect to the dual norms on W^*, V^*, whose operator norm is less than or equal to the operator norm of T. The operator norms of T and T^* are actually the same, because of the characterization (39.1) of the operator norm of T.

For example, if $V = W$ and T is the identity mapping on V, then T^* is the identity mapping on V^*. In the previous section, we saw that inclusion and quotient mappings are dual to each other.

Let V_1, V_2, and V_3 be vector spaces, all real or all complex complex, and equipped with norms. Also let $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ be bounded linear mappings, so that $T_2 \circ T_1$ is a bounded linear mapping from V_1 into V_3. It follows easily from the definitions that

$$(T_2 \circ T_1)^* = T_1^* \circ T_2^* \tag{39.3}$$

as bounded linear mappings from V_3^* into V_1^*.

40 Second duals

Let V be a real or complex vector space with a norm $\|\cdot\|$, let V^* be its dual space with the dual norm $\|\cdot\|_*$, and let V^{**} be the dual of V^* with its dual norm $\|\cdot\|_{**}$. As before, each $v \in V$ determines a bounded linear functional η_v on V^*, defined by

$$\eta_v(\lambda) = \lambda(v) \tag{40.1}$$

for each $\lambda \in V^*$, and the linear mapping $v \mapsto \eta_v$ is an isometric embedding from V into V^{**}. Note that the weak topology on V corresponds exactly to the topology induced on the copy of V in V^{**} by the weak* topology on V^{**} as the dual of V^*. A Banach space V is said to be reflexive if this embedding maps V onto V^{**}, and finite-dimensional spaces are automatically reflexive.

Let $\lambda_1, \ldots, \lambda_n$ be finitely many bounded linear functionals on V, and consider

$$W = \{v \in V : \lambda_1(v) = \cdots = \lambda_n(v) = 0\}. \tag{40.2}$$

This is a closed linear subspace of V of codimension $\le n$, which is to say that

$$\dim V/W \le n. \tag{40.3}$$

The space $W^\perp$ of bounded linear functionals μ on V that vanish on W is spanned by $\lambda_1, \ldots, \lambda_n$, and can be identified with $(V/W)^*$, as in Section 38.

Let ϕ be a bounded linear functional on V^*, and let ψ be its restriction to $W^\perp$. Since V/W is reflexive, ψ can be expressed by evaluation at an element of V/W whose quotient norm is equal to the norm of ψ as a linear functional on $W^\perp$, which is less than or equal to $\|\phi\|_{**}$. It follows from the definition of the quotient norm that for each $\epsilon > 0$ there is a $v \in V$ such that

$$\|v\| < \|\phi\|_{**} + \epsilon \tag{40.4}$$

and $\phi(\mu) = \mu(v)$ for each $\mu \in W^\perp$. In particular, the copy of V in V^{**} is dense in V^{**} with respect to the weak* topology on V^{**} as the dual of V^*, with control on the norms involved in the approximation of elements of V^{**} by elements of the copy of V.

For example, let E be an infinite set, and let V be $c_0(E)$. Thus V^* can be identified with $\ell^1(E)$, and V^{**} can be identified with $\ell^\infty(E)$. The natural embedding of V into V^{**} corresponds to the obvious inclusion of $c_0(E)$ in $\ell^\infty(E)$. In this case, the approximation of elements of $\ell^\infty(E)$ by elements of $c_0(E)$ can be seen quite concretely. It is a bit more convenient to think in terms of approximating $f \in \ell^\infty(E)$ by functions of the form f_A, where $A \subseteq E$ has only finitely many elements, $f_A = f$ on A, and $f_A = 0$ on $E \backslash A$. These approximations satisfy

$$\|f_A\|_\infty \le \|f\|_\infty \tag{40.5}$$

automatically. If $h_1, \ldots, h_n$ are summable functions on E, then

$$\Big|\sum_{x \in E} f(x)\, h_j(x) - \sum_{x \in E} f_A(x)\, h(x)\Big| \tag{40.6}$$

can be made arbitrarily small for $j = 1, \ldots, n$, by making

$$\|f\|_\infty \sum_{x \in E \backslash A} |h_j(x)| \tag{40.7}$$

arbitrarily small.

41 Continuous functions

Let X be a locally compact Hausdorff topological space, and let

$$\mathcal{C}(X, \mathbf{R}), \quad \mathcal{C}(X, \mathbf{C}) \tag{41.1}$$

be the vector spaces of continuous real or complex-valued functions on X. If K is a nonempty compact set in X, then

$$\|f\|_K = \sup_{x \in K} |f(x)| \tag{41.2}$$

defines a seminorm on these vector spaces. The collection of these seminorms satisfies the positivity condition that for each nonzero continuous function f on X there is a nonempty compact set $K \subseteq X$ such that $\|f\|_K \neq 0$, and thus determines a nice topology on these vector spaces, as in Section 24. A sequence $\{f_j\}_{j=1}^\infty$ of continuous functions on X converges to another continuous function f on X in this topology if and only if $\{f_j\}_{j=1}^\infty$ converges to f uniformly on compact subsets of X.

If λ is a bounded linear functional on one of these vector spaces with respect to this collection of seminorms, then there is a nonempty compact set $K \subseteq X$ and an $A \geq 0$ such that

$$|\lambda(f)| \leq A \, \|f\|_K \tag{41.3}$$

for every continuous function f on X. This uses the fact that the union of finitely many compact subsets of X is a compact set to reduce finitely many seminorms to a single seminorm in this situation. In particular,

$$\lambda(f) = 0 \tag{41.4}$$

when $f = 0$ on K.

Suppose that X is also σ-compact, so that there is a sequence of nonempty compact subsets $K_1, K_2, \ldots$ of X such that

$$X = \bigcup_{j=1}^\infty K_j. \tag{41.5}$$

We may as well ask in addition that

$$K_l \subseteq K_{l+1} \tag{41.6}$$

for each $l \geq 1$, since otherwise we can replace K_l with

$$\bigcup_{j=1}^l K_j \tag{41.7}$$

for each l. Using the fact that X is locally compact, one can improve this to get that

$$K_l \subseteq K_{l+1}^\circ \tag{41.8}$$

for each l, where E° denotes the interior of a set $E \subseteq X$. More precisely, local compactness implies that any compact set in X is contained in the interior of another compact set, and this can be applied repeatedly to get the previous condition. In particular,

$$X = \bigcup_{j=1}^{\infty} K_j^\circ. \tag{41.9}$$

If $K \subseteq X$ is compact, then it follows that

$$K \subseteq K_j^\circ \subseteq K_j \tag{41.10}$$

for some j. Indeed, K is contained in the union of finitely many K_j°'s, by compactness, and therefore in a single K_j°, by monotonicity. In this case,

$$\|f\|_K \le \|f\|_{K_j} \tag{41.11}$$

for each continuous function f on X. This shows that the sequence of seminorms corresponding to this sequence of compact sets suffices to determine the same topology on the space of continuous functions on X. For example, if $X = \mathbf{R}^n$, then one can take K_j to be the closed ball in $\mathbf{R}^n$ with respect to the standard metric centered at the origin and with radius j. Of course, if X is compact, then it suffices to take $K = X$, for which the corresponding seminorm is the supremum norm.

If X is equipped with the discrete topology, then every function on X is continuous, and only the finite subsets of X are compact. In this case, X is σ-compact if and only if X has only finitely or countably many elements. For any X, it is easy to see that the continuous functions on X with compact support are dense among all continuous functions on X with respect to the topology described above. Indeed, if f is any continuous function on X and $K \subseteq X$ is compact, then there is a continuous function g on X with compact support such that $f = g$ on K. This follows from the fact that there is a continuous function h on X with compact support such that $h = 1$ on K, so that $g = f\,h$ has compact support and is equal to f on K.

42 Bounded sets

Let W be a real or complex vector space. A set $E \subseteq W$ is said to be *bounded* with respect to a seminorm N on W if $N(w)$ is bounded for $w \in E$. Similarly, E is bounded with respect to a collection $\mathcal{N}$ of seminorms on W if E is bounded with respect to each $N \in \mathcal{N}$.

Let V be a real or complex vector space equipped with a norm $\|v\|$, let V^* be the dual space of bounded linear functionals on V with the dual norm $\|\lambda\|_*$, and consider the collection of seminorms on V^* given by

$$N_v^*(\lambda) = |\lambda(v)|, \ v \in V. \tag{42.1}$$

Any bounded set in V^* with respect to the dual norm is automatically bounded with respect to this collection of seminorms. Conversely, if V is a Banach space, then any bounded set in V^* with respect to this collection of seminorms is a bounded set with respect to the dual norm, as in Section 30. In the same way, a set in V is bounded with respect to the norm if and only if it is bounded with respect to the seminorms

$$N_\lambda(v) = |\lambda(v)|, \ \lambda \in V^*. \tag{42.2}$$

Let X be a locally compact Hausdorff topological space, and consider the vector space W of continuous real or complex-valued functions on X, as in the previous section. A set $E \subseteq W$ is bounded with respect to the supremum seminorms over compact subsets of X if and only if for each nonempty compact set $K \subseteq X$ there is a nonnegative real number $C(K)$ such that

$$|f(x)| \le C(K) \text{ for every } f \in E \text{ and } x \in K. \tag{42.3}$$

This does not mean that the functions $f \in E$ have to be uniformly bounded on X, since $C(K)$ depends on K.

The *convex hull* $\widehat{E}$ of a set E in a real or complex vector space W consists of all finite sums of the form

$$\sum_{j=1}^{n} t_j\, w_j, \tag{42.4}$$

where $t_1, \ldots, t_n$ are nonnegative real numbers such that $\sum_{j=1}^{n} t_j = 1$, and $w_1, \ldots, w_n \in E$. Thus $\widehat{E}$ is a convex set, $E \subseteq \widehat{E}$, and $\widehat{E}$ is contained in any convex set that contains E. If E is bounded with respect to a seminorm N on W, then it is easy to see that $\widehat{E}$ is also bounded with respect to N. Hence $\widehat{E}$ is bounded with respect to a collection $\mathcal{N}$ of seminorms on W when E is bounded with respect to $\mathcal{N}$.

43 Hahn–Banach, revisited

Let V be a real or complex vector space, let W be a linear subspace of V, and let λ be a linear functional on W. Suppose that there is a nonnegative real number L and a seminorm N on V such that

$$|\lambda(w)| \le L\, N(w) \tag{43.1}$$

for every $w \in W$. Under these conditions, there is an extension of λ to a linear functional μ on V such that

$$|\mu(v)| \le L\, N(v) \tag{43.2}$$

for every $v \in V$. This is the same as the earlier formulation of the Hahn-Banach theorem, except that N is allowed to be a seminorm instead of a norm. It is easy to see that the same arguments work in this case. Alternatively, if

$$Z = \{z \in V : N(z) = 0\}, \tag{43.3}$$

then Z is a linear subspace of V. The intersection of W and Z is contained in the kernel of λ, by (43.1), and one can first extend λ to the linear span of W and Z by setting equal to 0 on Z. This permits the problem to be transferred to the quotient space V/Z, on which N becomes a norm in an obvious way.

Suppose that $\mathcal{N}$ is a collection of seminorms on V that satisfies the usual positivity condition, so that for each $v \in V$ with $v \neq 0$ there is an $N \in \mathcal{N}$ such that $N(v) > 0$. Let V^* be the vector space of continuous linear functionals on V with respect to the topology determined by $\mathcal{N}$. If $\mathcal{N}$ consists of a single norm on V, then this is the same as the dual space defined in Section 14.

Let u be a nonzero vector in V, and fix $N \in \mathcal{N}$ such that $N(u) > 0$. If W is the 1-dimensional linear subspace of V spanned by u, and λ is the linear functional on W that satisfies $\lambda(u) = N(u)$, then (43.1) holds with $L = 1$. Hence there is an extension of λ to a linear functional μ on V such that (43.2) holds with $L = 1$. In particular, $\mu \in V^*$ and $\mu(u) \neq 0$.

Now let W_0 be a closed linear subspace of V, and let u be a vector in V not in W_0. By the definition of the topology on V determined by $\mathcal{N}$, there are finitely many seminorms $N_1, \ldots, N_l \in \mathcal{N}$ and a positive real number r such that

$$\max(N_1(u - w), \ldots, N_l(u - w)) \geq r \tag{43.4}$$

for every $w \in W_0$. Put

$$N(v) = \max(N_1(v), \ldots, N_l(v)), \tag{43.5}$$

which is also a seminorm on V. Let W be the linear subspace of V spanned by W_0 and u, and let λ be the linear functional on W defined by

$$\lambda(t\,u + w) = t\,r \tag{43.6}$$

for $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, and $w \in W_0$. Thus (43.4) implies that

$$|\lambda(t\,u + w)| \leq N(t\,u + w) \tag{43.7}$$

for every $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, and $w \in W_0$. This is the same as saying that (43.1) holds with $L = 1$. It follows that there is an extension of λ to a linear functional μ on V that satisfies (43.2) with $L = 1$.

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