An introduction to some aspects of functional analysis, 3: Topological vector spaces

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Abstract

In these notes, we give an overview of some aspects of topological vector spaces, including the use of nets and filters.

Contents

1 Basic notions

Let V be a vector space over the real or complex numbers, and suppose that V is also equipped with a topological structure. In order for V to be a topological vector space, we ask that the topological and vector spaces structures on V be compatible with each other, in the sense that the vector space operations be continuous mappings. More precisely, this means that addition of vectors in V should be continuous as a mapping from $V \times V$ into V, where $V \times V$ is equipped with the product topology associated to the given topology on V. Similarly, scalar multiplication should be continuous as a mapping from $\mathbf{R} \times V$ or $\mathbf{C} \times V$ into V, as appropriate. This uses the product topology on $\mathbf{R} \times V$ or $\mathbf{C} \times V$ associated to the standard topology on $\mathbf{R}$ or $\mathbf{C}$ and the given topology on V. Some authors include the additional condition that $\{0\}$ be a closed set in V, and we shall follow this convention here as well. Note that a topological vector space is automatically a commutative topological group with respect to addition, where the latter is defined analogously in terms of the continuity of the group operations.

Let V and W be topological vector spaces, both real or both complex. In this context, we are especially interested in mappings from V into W which are both continuous and linear. In particular, a mapping $T : V \to W$ is said to be an isomorphism from V onto W as topological vector spaces if it is an isomorphism from V onto W both as vector spaces and topological spaces. Equivalently, T should be a one-to-one linear mapping from V onto W which is a homeomorphism, so that both T and its inverse are continuous linear mappings.

If n is a positive integer, then $\mathbf{R}^n$ and $\mathbf{C}^n$ are topological vector spaces with respect to their standard vector space and topological structures. If V is an n-dimensional real or complex vector space, then V is isomorphic to $\mathbf{R}^n$ or $\mathbf{C}^n$ as a vector space, as appropriate. Let T be such an isomorphism, which is to say a one-to-one linear mapping from $\mathbf{R}^n$ or $\mathbf{C}^n$ onto V. We can also define a topology on V so that T is a homeomorphism, in which case V becomes a topological vector space isomorphic to $\mathbf{R}^n$ or $\mathbf{C}^n$.

Suppose now that $\widetilde{T}$ is another one-to-one linear mapping from $\mathbf{R}^n$ or $\mathbf{C}^n$ onto V, as appropriate. Thus $T^{-1} \circ \widetilde{T}$ is a one-to-one linear mapping from $\mathbf{R}^n$ or $\mathbf{C}^n$ onto itself. Every linear mapping from $\mathbf{R}^n$ or $\mathbf{C}^n$ into itself is continuous, and hence $T^{-1} \circ \widetilde{T}$ is a homeomorphism on $\mathbf{R}^n$ or $\mathbf{C}^n$. This implies that the topology on V described in the previous paragraph does not depend on the choice of isomorphism T. It can be shown that any n-dimensional topological vector space over the real or complex numbers is isomorphic as a topological vector space to $\mathbf{R}^n$ or $\mathbf{C}^n$ with its standard topology, but this is more complicated.

2 Translations and dilations

Let V be a topological vector space over the real or complex numbers. If $a \in V$, then let T_a be the mapping from V into itself defined by

$$T_a(v) = a + v. \tag{2.1}$$

It follows easily from the continuity of addition on V that T_a is a continuous mapping from V into itself for each $a \in V$. Moreover, T_a is a one-to-one mapping from V onto itself, with inverse given by T_{-a}, which is also continuous, and hence T_a is a homeomorphism from V onto itself for each $a \in V$. In particular, the hypothesis that $\{0\}$ be a closed set in V implies that $\{a\}$ is a closed set in V for every $a \in V$, so that V satisfies the first separation condition as a topological space.

Similarly, if t is a real or complex number, as appropriate, then

$$v \mapsto t\,v \tag{2.2}$$

defines a continuous mapping from V into itself, by continuity of scalar multiplication. If $t \neq 0$, then this is a homeomorphism on V, because the inverse mapping corresponds to multiplication by $1/t$.

If $a \in V$ and $B \subseteq V$, then put

$$a + B = T_a(B) = \{a + w : w \in B\}. \tag{2.3}$$

One can define $B + a$ in the same way, which is equal to $a + B$, because of commutativity of addition. If $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, then put

$$t\,B = \{t\,w : w \in B\}. \tag{2.4}$$

Of course, this reduces to

$$-B = \{-w : w \in B\} \tag{2.5}$$

when $t = -1$. We may also use the notation $a - B$ for $a + (-B)$, etc.

If $A, B \subseteq V$, then put

$$A + B = \{v + w : v \in A, w \in B\}. \tag{2.6}$$

Equivalently,

$$A + B = \bigcup_{a \in A} (a + B) = \bigcup_{b \in B} (A + b). \tag{2.7}$$

Note that $A + B$ is an open set in V when either A or B is an open set.

3 Separation conditions

Let V be a topological vector space over the real or complex numbers. Also let $w \in V$ with $w \neq 0$ be given, and put $W = V \backslash \{w\}$. Thus W is an open set in

V that contains 0, and continuity of addition on V at 0 implies that there are open subsets U_1, U_2 of V that contain 0 and satisfy

$$U_1 + U_2 \subseteq W. \tag{3.1}$$

The latter implies that $u_1 + u_2 \neq w$ for every $u_1 \in U_1$ and $u_2 \in U_2$, and hence that $u_1 \neq w - u_2$ for every $u_1 \in U_1$ and $u_2 \in U_2$. Therefore

$$U_1 \cap (w - U_2) = \emptyset. \tag{3.2}$$

Note that $w - U_2$ is an open set in V that contains w, by the remarks in the previous section. It follows that V is Hausdorff, since one can use translations to reduce an arbitrary pair v, w of distinct elements of V to the case where $v = 0$.

Now let E be a closed set in V that does not contain 0, and put $W = V \backslash E$. As in the previous paragraph, W is an open set in V that contains 0, and continuity of addition at 0 implies that there are open sets U_1, U_2 containing 0 and satisfying (3.1). This implies that

$$U_1 \cap (E - U_2) = \emptyset, \tag{3.3}$$

where $E - U_2 = E + (-U_2)$ is an open set in V that contains E as a subset. It follows that V is regular, or satisfies the third separation condition as a topological space, since one can use translations again to reduce an arbitary element v of $V \backslash E$ to the case where $v = 0$.

However, one can have disjoint closed sets A, B such that

$$(A + U_1) \cap (B - U_2) \neq \emptyset \tag{3.4}$$

for any pair U_1, U_2 of open sets containing 0. For example, one can take $V = \mathbf{R}$ with the standard topology, A to be the set $\mathbf{Z}_+$ of positive integers, and

$$B = \{j + (1/j) : j \in \mathbf{Z}_+\}. \tag{3.5}$$

Alternatively, one can take $V = \mathbf{R}^2$ with the standard topology, and A, B to be the hyperbolae

$$A = \{(x, y) \in \mathbf{R}^2 : x\, y = 1\}, \tag{3.6}$$

$$B = \{(x, y) \in \mathbf{R}^2 : x\, y = -1\}. \tag{3.7}$$

In both cases, A, B are disjoint closed sets with elements $v \in A$, $w \in B$ such that $v - w$ is contained in arbitrarily small neighborhoods of 0.

If V is any topological vector space again, $E \subseteq V$ is a closed set, $K \subseteq V$ is compact, and $E \cap K = \emptyset$, then there are open sets $U', U'' \subseteq V$ containing 0 such that

$$(K + U') \cap (E - U'') = \emptyset. \tag{3.8}$$

To see this, we can apply the earlier argument to get for each $p \in K$ open subsets $U_1(p)$, $U_2(p)$ of V containing 0 and satisfying

$$(p + U_1(p)) \cap (E - U_2(p)) = \emptyset. \tag{3.9}$$

We can also use continuity of addition at 0 again to get for each $p \in K$ open subsets $U_{1,1}(p)$, $U_{1,2}(p)$ of V containing 0 and satisfying

$$U_{1,1}(p) + U_{1,2}(p) \subseteq U_1(p). \tag{3.10}$$

Because $p+U_{1,1}(p)$ is an open set in V containing p for each $p \in K$, compactness of K implies that there are finitely many elements $p_1, \ldots, p_n$ of K such that

$$K \subseteq \bigcup_{j=1}^{n} (p_j + U_{1,1}(p_j)). \tag{3.11}$$

Put $U' = \bigcap_{j=1}^{n} U_{1,2}(p_j)$ and $U'' = \bigcap_{j=1}^{n} U_2(p_j)$, which are each open subsets of V that contain 0, since they are intersections of finitely many open subsets of V containing 0. Observe that

$$\begin{aligned} K + U' &\subseteq \bigcup_{j=1}^{n} (p_j + U_{1,1}(p_j) + U') \\ &\subseteq \bigcup_{j=1}^{n} (p_j + U_{1,1}(p_j) + U_{1,2}(p_j)) \\ &\subseteq \bigcup_{j=1}^{n} (p_j + U_1(p_j)). \end{aligned} \tag{3.12}$$

Using this and (3.9), it is easy to check that (3.8) holds with these choices of U' and U''.

4 Bounded sets

Let V be a topological vector space over the real or complex numbers, and let E be a subset of V. We say that E is *bounded* in V if it satisfies one of the following four equivalent conditions. (1) For each open set $U \subseteq V$ with $0 \in U$ there is a $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, such that $E \subseteq t\,U$. (2) For each open set $U \subseteq V$ with $0 \in U$ there is a $t \geq 0$ such that $E \subseteq t\,U$. (3) For each open set $U \subseteq V$ with $0 \in U$ there is an $r \geq 0$ such that $E \subseteq t\,U$ when $t > r$. (4) For each open set $U \subseteq V$ with $0 \in U$ there is an $r \geq 0$ such that $E \subseteq t\,U$ when $|t| > r$. Note that r is implicitly a real number in conditions (3) and (4), t is implicitly real in conditions (2) and (3), while t may be real or complex, as appropriate, in condition (4).

Of course, each of these conditions appears to be stronger than the previous one. To show that they are equivalent, it suffices to prove that (1) implies (4). To do this, the main point is to use continuity of scalar multiplication at 0. Let $U \subseteq V$ be an arbitrary open set with $0 \in U$. Because of continuity of scalar multiplication, there are an open set $W \subseteq V$ with $0 \in W$ and a positive real number δ such that $t\,w \in U$ for every $w \in W$ and $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, with $|t| < \delta$. Here $|t|$ denotes the absolute value of t when $t \in \mathbf{R}$, and the modulus of t when $t \in \mathbf{C}$. Equivalently, $t\,W \subseteq U$ when $|t| < \delta$, and hence

$$W \subseteq t\,U \tag{4.1}$$

when $|t| > 1/\delta$. If E satisfies the boundedness condition (1), then $E \subseteq r\,W$ for some $r \in \mathbf{R}$ or $\mathbf{C}$, as appropriate. This implies that $E \subseteq t\,U$ when $|t| > |r|/\delta$, as desired.

Suppose that A and B are bounded subsets of V, and let us check that $A \cup B$ is also bounded. If U is an open set in V that contains 0, then $A \subseteq t\,U$ when $|t|$ is sufficiently large, and similarly for B. This implies that $A \cup B \subseteq t\,U$ when $|t|$ is sufficiently large, as desired. Now let us show that $A + B$ is bounded as well. Let $U \subseteq V$ be an open set with $0 \in U$ again, and let U_1, U_2 be open subsets of V containing 0 such that $U_1 + U_2 \subseteq U$. As usual, we can get the existence of U_1, U_2 from the continuity of addition at 0 in V. The boundedness of A implies that $A \subseteq t\,U_1$ when $|t|$ is sufficiently large, and the boundedness of B implies that $B \subseteq t\,U_2$ when $|t|$ is sufficiently large. Hence

$$A + B \subseteq t\,U_1 + t\,U_2 \subseteq t\,U \tag{4.2}$$

when $|t|$ is sufficiently large, as desired.

If $E \subseteq V$ is bounded and $A \subseteq E$, then A is bounded too. Let us check that the closure $\overline{E}$ of E in V is bounded when E is bounded. To see this, let U be any open set in V that contains 0, and let W be another open set in V such that $0 \in W$ and $\overline{W} \subseteq U$. The existence of W follows from the regularity of V as a topological space. Because E is bounded, $E \subseteq t\,W$ when $|t|$ is sufficiently large, and hence

$$\overline{E} \subseteq t\,\overline{W} \subseteq t\,U \tag{4.3}$$

when $|t|$ is sufficiently large, as desired.

5 Norms

Let V be a vector space over the real or complex numbers. As usual, a norm on V is a nonnegative real-valued function $N(v)$ on V such that $N(v) = 0$ if and only if $v = 0$,

$$N(t\,v) = |t|\,N(v) \tag{5.1}$$

for every $v \in V$ and $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, and

$$N(v + w) \le N(v) + N(w) \tag{5.2}$$

for every $v, w \in V$. If $N(v)$ is a norm on V, then

$$d(v, w) = N(v - w) \tag{5.3}$$

defines a metric on V. It is well known and not too difficult to check that the topology on V associated to this metric satisfies the requirements of a topological vector space.

In this case, a set $E \subseteq V$ would normally be considered to be bounded if the norms of the elements of E have a finite upper bound. It is easy to see that this implies that E is bounded as a subset of V as a topological vector space, because every open set in V that contains 0 also contains an open ball around 0 with respect to the norm. Conversely, if E is bounded as a subset of V as a topological vector space, then one can check that E is bounded in the usual sense, by applying any of the conditions in the previous section with U equal to the open unit ball in V associated to the norm.

Remember that a set E in a vector space V is said to be *convex* if

$$t\, v + (1 - t)\, w \in E \tag{5.4}$$

for every $v, w \in E$ and $t \in \mathbf{R}$ with $0 \le t \le 1$. Suppose that N is a norm on V, and let

$$B = \{v \in V : N(v) < 1\} \tag{5.5}$$

be the open unit ball in V associated to V. It is easy to see that B is convex under these conditions, using the triangle inequality.

Conversely, suppose that $N(v)$ is a nonnegative real-valued function on V that satisfies the homogeneity condition (5.1). If the corresponding open unit ball B is convex, then N also satisfies the triangle inequality (5.2). To see this, let v, w be arbitrary vectors in V, and let a, b be real numbers such that $a > N(v)$, $b > N(w)$. Thus $v' = v/a$ and $w' = w/b$ are elements of B. Using the convexity of B with $t = a/(a + b)$, so that $1 - t = b/(a + b)$, we get that

$$\frac{v + w}{a + b} = \frac{a}{a + b}\, v' + \frac{b}{a + b}\, w' \in B. \tag{5.6}$$

Equivalently, $N((v + w)/(a + b)) < 1$, which is to say that

$$N(v + w) < a + b. \tag{5.7}$$

This implies (5.2), since $a > N(v)$ and $b > N(w)$ are arbitrary.

6 L^p Spaces

Let $(X, \mathcal{A}, \mu)$ be a measure space, so that X is a set, $\mathcal{A}$ is a σ-algebra of "measurable" subsets of X, and μ is a nonnegative measure defined on $\mathcal{A}$. Basic examples include Lebesgue measure on the unit interval or real line, and counting measure on any set. If p is a positive real number, then $L^p(X)$ denotes the space of measurable real or complex-valued functions f on X such that $|f|^p$ is integrable on X, and we put

$$\|f\|_p = \Big(\int_X |f|^p \, d\mu\Big)^{1/p}. \tag{6.1}$$

More precisely, it is customary to identify functions on X that are equal almost everywhere, since $\|f\|_p = 0$ if and only if $f = 0$ almost everywhere on X. One can extend this to $p = \infty$ by taking $L^\infty(X)$ to be the space of measurable real or complex-valued functions that are "essentially bounded", and taking $\|f\|_\infty$ to be the "essential supremum" of $|f|$.

With these conventions, it is well known that $L^p(X)$ is a vector space when $1 \le p \le \infty$, and that $\|f\|_p$ defines a norm on $L^p(X)$. If $0 < p < 1$, then $L^p(X)$ is still a vector space, but $\|f\|_p$ is not necessarily a norm, because it may not satisfy the triangle inequality. As a concrete example, let X be a set with exactly two elements, equipped with counting measure. The vector space V of real-valued functions on X may be identified with $\mathbf{R}^2$, in which case $\|f\|_p$ corresponds to

$$\|v\|_p = (|v_1|^p + |v_2|^p)^{1/p} \tag{6.2}$$

when $0 < p < \infty$, and to

$$\|v\|_\infty = \max(|v_1|, |v_2|) \tag{6.3}$$

when $p = \infty$. It is easy to see that $\|v\|_p$ does not satisfy the triangle inequality on $\mathbf{R}^2$ when $0 < p < 1$, and one can also observe that the corresponding open unit ball is not convex.

To deal with this, let us check that

$$(a + b)^p \le a^p + b^p \tag{6.4}$$

for any nonnegative real numbers a, b when $0 < p \le 1$. Indeed,

$$\max(a, b) \le (a^p + b^p)^{1/p}, \tag{6.5}$$

and hence

$$\begin{aligned} a + b &\le \max(a, b)^{1-p} \, (a^p + b^p) \\ &\le (a^p + b^p)^{(1-p)/p+1} = (a^p + b^p)^{1/p}. \end{aligned} \tag{6.6}$$

This implies (6.4), by taking pth powers of both sides.

If $f, g \in L^p(X)$, $0 < p \le 1$, then

$$\|f + g\|_p^p = \int_X |f + g|^p \, d\mu \le \int_X |f|^p \, d\mu + \int_X |g|^p \, d\mu = \|f\|_p^p + \|g\|_p^p \tag{6.7}$$

by (6.4). This implies that

$$d_p(f, g) = \|f - g\|_p^p \tag{6.8}$$

defines a metric on $L^p(X)$, and one can check that $L^p(X)$ is a topological vector space with respect to the topology associated to this metric. One can also check that a set $E \subseteq L^p(X)$ is bounded as a subset of $L^p(X)$ as a topological vector space if and only if E is bounded in the usual sense with respect to $\|f\|_p$, as in the previous section.

7 Balanced sets

Let V be a vector space over the real or complex numbers, not necesarily with a topology for the moment. A set $A \subseteq V$ is said to be *balanced* if

$$t\,A \subseteq A \tag{7.1}$$

for every $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, with $|t| \le 1$. This implies that $0 \in A$ when $A \neq \emptyset$, and the condition that (7.1) holds when t is a real number such that $0 \le t \le 1$ is the same as saying that A is star-like about 0. If $t \in \mathbf{R}$ or $\mathbf{C}$ satisfies $|t| = 1$, then we can apply (7.1) to t and to t^{-1} to get that

$$t\,A = A. \tag{7.2}$$

Of course, this is trivial when $t = 1$, and when $t = -1$ this reduces to the condition that A be symmetric in the sense that $A = -A$.

Now let V be a topological vector space, and let U be an open set in V that contains 0. Using the continuity of scalar multiplication at 0 as before, we get that there are an open set W in V that contains 0 and a $\delta > 0$ such that $t\,W \subseteq U$ for each $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, with $|t| < \delta$. Put

$$W_1 = \bigcup_{0<|t|<\delta} t\,W, \tag{7.3}$$

where more precisely the union is taken over all $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, such that $0 < |t| < \delta$. Thus

$$W_1 \subseteq U, \tag{7.4}$$

W_1 is balanced by construction, and W_1 is an open set in V that contains 0 because it is a union of such sets.

This shows that every open set $U \subseteq V$ with $0 \in V$ contains a nonempty balanced open set. Equivalently, there is a local base for the topology of V at 0 consisting of balanced open sets. In particular, in order to check that a set $E \subseteq V$ is bounded, it suffices to consider nonempty balanced open sets $U \subseteq V$ instead of arbitrary open sets that contain 0. In this case, the four variants of boundedness mentioned in Section 4 are all obviously the same. Of course, this is very similar to the earlier arguments, which also used the continuity of scalar multiplication.

If N is a norm on V, then the open ball in V corresponding to N centered at 0 and with any positive radius is clearly balanced, because of the homogeneity property of the norm. This also works in L^p spaces with $\|f\|_p$ in place of the norm, even when $0 < p < 1$, so that $\|f\|_p$ may not be a norm. In both cases, this implies directly that any open set U in the corresponding topology that contains 0 also contains a nonempty balanced open set, because U contains an open ball around 0.

8 The absorbing property

Let V be a topological vector space over the real or complex numbers, and let U be an open set in V that contains 0. If v is any vector in V, then $0\,v = 0 \in U$, and hence there is a positive real number $\delta(v)$ such that

$$t\,v \in U \tag{8.1}$$

for every $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, with $|t| < \delta(v)$. This uses the continuity of scalar multiplication at v and $t = 0$. Subsets of V with this property are often said to be *absorbing*, although this term is also sometimes used for weaker versions of this condition. These different versions are fortunately the same for balanced sets, with which one may be especially concerned.

Using the absorbing property for open sets that contain 0, it is easy to see that finite subsets of V are bounded. Let us check that every compact set $K \subseteq V$ is also bounded. As in the previous section, it suffices to check the boundedness condition for a nonempty balanced open set $U \subseteq V$. The absorbing property implies that

$$\bigcup_{n=1}^{\infty} n\,U = V, \tag{8.2}$$

and hence K is contained in the union of $n\,U$ for finitely many positive integers n, by compactness. If U is balanced, then $j\,U \subseteq l\,U$ when $j \le l$, and it follows that $K \subseteq n\,U$ for sufficiently large n, as desired.

Let $N(v)$ be a nonnegative real-valued function on a real or complex vector space V that satisfies the same homogeneity property as a norm, which is to say that $N(t\,v) = |t|\,N(v)$ for every $v \in V$ and $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate. Thus $N(0) = 0$, as one can see by taking $t = 0$. Let r be a positive real number, and put

$$B_N(r) = \{v \in V : N(v) < r\}. \tag{8.3}$$

This is clearly a balanced set in V, and it has the absorbing property that

$$t\,v \in B_N(r) \tag{8.4}$$

when $|t|\,N(v) < r$.

9 Seminorms

Let V be a vector space over the real or complex numbers. A nonnegative real-valued function $N(v)$ is said to be a *seminorm* on V if

$$N(t\,v) = |t|\,N(v) \tag{9.1}$$

for every $v \in V$ and $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, and

$$N(v + w) \le N(v) + N(w) \tag{9.2}$$

for every $v, w \in V$. Thus a seminorm is the same as a norm, but without the positivity condition that $N(v) > 0$ when $v \neq 0$. Note that $N(0) = 0$, by taking $t = 0$ in the homogeneity condition. If $w \in V$ and $r > 0$, then the open ball in V with center w and radius r associated to N may be defined as usual by

$$B_N(w, r) = \{v \in V : N(v - w) < r\}. \tag{9.3}$$

It is easy to see that this is a convex set in V for every $w \in V$ and $r > 0$, using the triangle inequality. Conversely, if $N(v)$ is a nonnegative real-valued function on V that satisfies the homogeneity condition (9.1), and if the corresponding unit ball $B_N(0, 1)$ is convex, then $N(v)$ also satisfies the triangle inequality (9.2), and hence is a seminorm, for the same reasons as in Section 5.

Now let $\mathcal{N}$ be a collection of seminorms on V. Using $\mathcal{N}$, we can define a topology on V, by saying that a set $U \subseteq V$ is an open set if for every $u \in U$ there are finitely many seminorms $N_1, \ldots, N_l$ in $\mathcal{N}$ and finitely many positive real numbers $r_1, \ldots, r_l$ such that

$$\bigcap_{j=1}^{l} B_{N_j}(u, r_j) \subseteq U. \tag{9.4}$$

It is easy to see that this defines a topology on V, by construction, and that every open ball in V with respect to every element of $\mathcal{N}$ is an open set with respect to this topology. The last part uses the triangle inequality, and is analogous to the fact that open balls are open sets in metric spaces. Equivalently, one can get a sub-base for the topology on V associated to $\mathcal{N}$ using open balls corresponding to arbitrary elements of $\mathcal{N}$.

Let us say that $\mathcal{N}$ is *nice* if for each $v \in V$ with $v \neq 0$ there is an $N \in \mathcal{N}$ such that $N(v) > 0$. Using this, one can check that $\{0\}$ is a closed set in V with respect to the topology associated to $\mathcal{N}$, and that V is Hausdorff and even regular with respect to this topology. One can also check that the vector space operations on V are continuous with respect to this topology, so that V becomes a topological vector space. If $\mathcal{N}$ consists of a single element N, then the niceness of $\mathcal{N}$ means exactly that N is a norm, and the topology on V associated to $\mathcal{N}$ reduces to the one determined by N as in Section 5.

Suppose that V is equipped with the topology determined by a nice collection of seminorms $\mathcal{N}$. In this case, a set $E \subseteq V$ is bounded as a subset of V as a topological space if and only if each seminorm $N \in \mathcal{N}$ is bounded on E. Indeed, if U is an open set in V that contains 0, then U contains the intersection of finitely many open balls centered at 0 corresponding to seminorms $N_1, \ldots, N_l \in \mathcal{N}$. If N_j is bounded on E for each $j = 1, \ldots, l$, then it is easy to see that $E \subseteq t\,U$ when $|t|$ is sufficiently large. Conversely, if E is bounded as a subset of V as a topological space and $N \in \mathcal{N}$, then it is easy to see that N is bounded on E, by applying the definition of boundedness to $U = B_N(0, 1)$.

10 An example

Let X be a nonempty set, and let V be the vector space of real or complex-valued functions on X, with respect to pointwise addition and scalar multiplication. If $x \in X$, then

$$N_x(f) = |f(x)| \tag{10.1}$$

defines a seminorm on V. The collection $\mathcal{N}$ of seminorms N_x on V, $x \in X$, is clearly a nice collection of seminorms on V, and hence defines a topology on V which makes V into a topological vector space. One can also think of V as the Cartesian product of a family of copies of $\mathbf{R}$ or $\mathbf{C}$ indexed by X, and the topology on V determined by $\mathcal{N}$ corresponds exactly to the product topology on V, associated to the standard topology on V. As in the previous section, a set $E \subseteq V$ is bounded as a subset of V as a topological vector space if and only if the restriction of N_x to E is bounded for each $x \in X$. Equivalently, this means that E is contained in the Cartesian product of a family of intervals in $\mathbf{R}$ or disks in $\mathbf{C}$, as appropriate. Using Tychonoff's theorem, it follows that closed and bounded subsets of V are compact.

11 Local convexity

A real or complex topological vector space V is said to be *locally convex* if there is a local base for the topology of V at 0 consisting of convex open sets. This means that for each open set U in V that contains 0 there is a convex open set W in V such that $0 \in W$ and $W \subseteq V$. This is also equivalent to saying that the convex open subsets of V form a base for the topology of V. If the topology of V is determined by a nice collection of seminorms, then V is locally convex, because ope balls associated to seminorms are convex. In particular, V is locally convex when the topology on V is determined by a norm.

Let A be a subset of a real or complex vector space V. The *convex hull* of A is the set $\mathrm{con}(A)$ consisting of all finite sums of the form

$$\sum_{j=1}^{n} t_j \, v_j, \tag{11.1}$$

where $v_1, \ldots, v_n \in A$, $t_1, \ldots, t_n \in \mathbf{R}$, $t_1, \ldots, t_n \geq 0$, and $\sum_{j=1}^{n} t_j = 1$. It is easy to see that $\mathrm{con}(A)$ is convex, $A \subseteq \mathrm{con}(A)$, and $A = \mathrm{con}(A)$ if and only if A is convex. If $E \subseteq V$ is convex and $A \subseteq E$, then $\mathrm{con}(A) \subseteq E$, so that the convex hull of A is the smallest convex set in V that contains A. If V is a locally convex topological vector space and $A \subseteq V$ is bounded, then it follows that the convex hull of A is bounded in V too. To see this, let U be an open set in V that contains 0, and let W be a convex open set in V such that $0 \in W$ and $W \subseteq U$. Because A is bounded, $A \subseteq t\,W$ when $|t|$ is sufficiently large. This implies that

$$\mathrm{con}(A) \subseteq t\,W \subseteq t\,U \tag{11.2}$$

for the same t's, since W is convex, as desired.

Suppose that $V = L^p(X)$ for some $0 < p < 1$, equipped with the topology discussed in Section 6. More precisely, suppose that X is the unit interval or the real line equipped with Lebesgue measure, for instance, or that X is an infinite set equipped with counting measure. If B is the open unit ball in $L^p(X)$, consisting of $f \in L^p(X)$ such that $\|f\|_p < 1$, then B is clearly bounded as a subset of $L^p(X)$ as a topological vector space. However, one can also check that the convex hull of B is not bounded in $L^p(X)$ under these conditions. In particular, $L^p(X)$ is not locally convex when $0 < p < 1$, at least if X is reasonably nice and nontrivial.

If V is a locally convex topological vector space, then it turns out that there is a nice collection of seminorms on V that determines the same topology. Without getting into the details, one can first show that there is a local base for the topology of V at 0 consisting of balanced convex open sets. If U is a nonempty balanced convex open subset of V, then one can show that there is a seminorm N_U on V such that the open unit ball in V associated to N_U is equal to U. More precisely, one can take N_U to be the Minkowski functional associated to U. One can then get a nice collection of seminorms on V that determines the same topology on V using the Minkowski functionals corresponding to balanced convex open subsets of V in a local base for the topology of V at 0.

12 Metrizability

If X is any topological space, then a necessary condition for the existence of a metric on X that determines the same topology is that for each $p \in X$ there be a countable local base for the topology of X at p. In the case of a topological vector space V, this necessary condition reduces to asking that there be a countable local base for the topology of V at 0, because of translation-invariance. A well-known theorem states that this simple necessary condition is actually sufficient for the metrizability of a topological vector space V over the real or complex numbers. More precisely, if there is a countable local base for the topology of V at 0, then it can be shown that there is a metric $d(v, w)$ on V which determines the same topology, and which is invariant under translations in the sense that

$$d(v + u, w + u) = d(v, w) \tag{12.1}$$

for every $u, v, w \in V$.

Of course, if the topology on V is defined by a norm $N(v)$, then we can take

$$d(v, w) = N(v - w), \tag{12.2}$$

as in Section 5. Suppose now that the topology on V is defined by a nice collection $\mathcal{N}$ of seminorms on V. If $\mathcal{N}$ consists of only finitely many seminorms $N_1, \ldots, N_l$, then we can take their maximum or sum to get a norm on V that defines the same topology, and we are back to the previous case. If instead $\mathcal{N}$ consists of an infinite sequence of seminorms $N_1, N_2, \ldots$, then we can get a translation-ivariant metric on V that determines the same topology on V, as follows. Put

$$d_j(v, w) = \min(N_j(v - w), 1/j) \tag{12.3}$$

for each positive integer j. This is a *semimetric* on V, which means that it is nonnegative, symmetric, and satisfies the triangle inequality, but it may be equal to 0 even when $v \neq w$. To get a metric on V, we put

$$d(v, w) = \max_{j \geq 1} d_j(v, w). \tag{12.4}$$

This is trivially equal to 0 when $v = w$, and otherwise it is easy to see that the maximum exists, because $d_j(v, w) \to 0$ as $j \to \infty$. It is also easy to see that $d(v, w)$ is symmetric and nonnegative, because of the corresponding properties of $d_j(v, w)$. If $v \neq w$, then $N_j(v - w) > 0$ for some j, since $\mathcal{N}$ is a nice collection of seminorms on V, so that $d(v, w) > 0$. Thus $d(v, w)$ is a metric on V, and it remains to check that the topology on V associated to $d(v, w)$ is the same as the one associated to $\mathcal{N}$. Note that an open ball with respect to $d(v, w)$ of any radius $r > 0$ is the same as the intersection of open balls with respect to the seminorms N_j with $r \leq 1/j$. There are only finitely many of these seminorms for each $r > 0$, and by taking r sufficiently small, one can get the intersection of open balls corresponding to $N_1, \ldots, N_l$ for any l. One also gets the intersection of open balls corresponding to $N_1, \ldots, N_l$ with arbitrarily small radii, by taking r sufficiently small, so that the topology on V defined by $\mathcal{N}$ is determined by $d(v, w)$ as well.

If the topology on V is determined by a nice collection $\mathcal{N}$ of seminorms, and if there is a countable local base for the topology of V at 0, then there is a nice subcollection $\mathcal{N}_1$ of $\mathcal{N}$ with only finitely or countably many elements that determines the same topology on V. This is because every open set in V containing 0 contains the intersection of open balls corresponding to finitely many elements of $\mathcal{N}$, so that only finitely or countably many elements of $\mathcal{N}$ are needed to get a local base for the topology of V at 0. Similarly, if V is a locally convex topological vector space with a countable local base for the topology of V at 0, then the topology of V may be described using only finitely or countably many seminorms. It suffices to use seminorms corresponding to balanced convex open subsets of V in a local base for the topology of V at 0, as in the previous section.

If $d(x, y)$ is a metric on any set X and t is a positive real number, then the minimum of $d(x, y)$ and t defines another metric $d'(x, y)$ on X that determines the same topology on X. Similarly, if $d(v, w)$ is a translation-invariant metric on a topological vector space V that determines the given topology on V, then the minimum $d'(v, w)$ of $d(v, w)$ and t defines another translation-invariant metric on V that determines the same topology on V. Of course, any set $E \subseteq V$ is bounded as a subset of V as a metric space with respect to $d'(v, w)$, while not every set $E \subseteq V$ is bounded as a subset of V as a topological vector space.

13 Continuous linear functionals

As usual, a linear functional λ on a real or complex vector space V is a linear mapping from V into the real or complex numbers, as appropriate. If V is a topological vector space and λ is continuous at 0, then there is an open set U in V such that $0 \in U$ and

$$|\lambda(u)| < 1 \tag{13.1}$$

for each $u \in U$. It follows that

$$|\lambda(v)| < \epsilon \tag{13.2}$$

for every $v \in \epsilon U$ and any positive real number ϵ, so that the existence of such an open set U characterizes the continuity of λ at 0. Note that continuity of λ at 0 implies that λ is continuous at every point in V, by linearity.

Suppose that the topology on V is determined by a nice collection of seminorms $\mathcal{N}$. If U is as in the previous paragraph, then there are finitely many seminorms $N_1, \ldots, N_l \in \mathcal{N}$ and positive real numbers $r_1, \ldots, r_l$ such that

$$\bigcap_{j=1}^{l} B_{N_j}(0, r_j) \subseteq U, \tag{13.3}$$

as in Section 9. This means that $|\lambda(v)| < 1$ when $v \in V$ satisfies $N_j(v) < r_j$ for $j = 1, \ldots, l$, or equivalently when $r_j^{-1} N_j(v) < 1$ for each j. This implies that

$$|\lambda(v)| \le \max_{1 \le j \le l} r_j^{-1} N_j(v) \tag{13.4}$$

for each $v \in V$, by muliplying v by positive real numbers and using the linearity of λ to reduce to the previous case.

Conversely, suppose that a linear functional λ on V satisfies

$$|\lambda(v)| \le C \max_{1 \le j \le l} N_j(v) \tag{13.5}$$

for some nonnegative real number C, finitely many seminorms $N_1, \ldots, N_l \in \mathcal{N}$, and every $v \in V$. In this case, it is easy to see that λ is continuous on V, by reversing the previous arguments. Of course, (13.4) implies (13.5), with C equal to the maximum of $r_1^{-1}, \ldots, r_l^{-1}$. Thus continuity of a linear functional λ on V is characterized by a condition like (13.5) when the topology on V is determined by a nice collection of seminorms.

If the topology on V is determined by a single norm N, then we can take $\mathcal{N} = \{N\}$ and $l = 1$ in (13.5), to get that a linear functional λ on V is continuous if and only if

$$|\lambda(v)| \le C\, N(v) \tag{13.6}$$

for some $C \ge 0$ and every $v \in V$. This also works on spaces like L^p spaces when $0 < p < 1$, where the topology is determined by a function that satisfies the same positivity and homogeneity properties as a norm, but might not be quite a norm itself. Thus, for any measure space X and $0 < p \le \infty$, a linear functional λ on $L^p(X)$ is continuous if and only if

$$|\lambda(f)| \le C \, \|f\|_p \tag{13.7}$$

for some $C \ge 0$ and every $f \in L^p(X)$.

14 The Hahn–Banach theorem

Let V be a vector space over the real or complex numbers, and let N be a seminorm on V. Suppose that W is a linear subspace of V, and that λ is a linear functional on W, such that

$$|\lambda(w)| \le C \, N(w) \tag{14.1}$$

for some $C \ge 0$ and every $w \in W$. Under these conditions, the theorem of Hahn and Banach implies that there is an extension of λ to a linear functional on V that satisfies (14.1) for every $w \in V$, with the same constant C. If N is a norm on V, then this condition corresponds to the continuity of λ on V with respect to the topology determined by N, as in the previous section. Otherwise, if $\mathcal{N}$ is a nice collection of seminorms on V, then N might be an element of $\mathcal{N}$, or the maximum of finitely many elements of $\mathcal{N}$, and again this condition implies the continuity of λ with respect to the topology associated to $\mathcal{N}$.

As an application, suppose that $\mathcal{N}$ is a nice collection of seminorms on V, $v \in V$, and $v \ne 0$, and let N be an element of $\mathcal{N}$ such that $N(v) > 0$. Let λ be the linear functional on the 1-dimensional linear subspace W of V spanned by v defined by

$$\lambda(t \, v) = t \, N(v) \tag{14.2}$$

for each $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, so that λ satisfies the previous condition with $C = 1$. The Hahn–Banach theorem implies that λ can be extended to a linear functional on all of V which satisfies the same condition, and which is therefore contionuous with respect to the topology on V associated to $\mathcal{N}$. This argument can be applied to any locally convex topological vector space V, to get that for each $v \in V$ with $v \ne 0$ there is a continuous linear functional λ on V such that $v \ne 0$. By contrast, if X is the unit interval equipped with Lebesgue measure and $0 < p < 1$, then one can show that the only continuous linear functional on $L^p(X)$ is the trivial one, with $\lambda(f) = 0$ for every $f \in L^p(X)$.

As another application, suppose that $\mathcal{N}$ is a nice collection of seminorms on V again, and let W_0 be a linear subspace of V which is closed with respect to the topology determined by $\mathcal{N}$. If $v \in V \backslash W_0$, then there is a linear functional λ on V which is continuous with respect to the topology determined by $\mathcal{N}$ such that $\lambda(w) = 0$ for every $w \in W_0$ and $\lambda(v) \ne 0$. It is easy to define λ initially on the linear subspace W of V spanned by W_0 and v, by setting

$$\lambda(w + t \, v) = t \tag{14.3}$$

for every $w \in W_0$ and $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate. Because W_0 is closed with respect to the topology determined by $\mathcal{N}$ and $v \in V \backslash W_0$, there are finitely many seminorms $N_1, \ldots, N_l$ in $\mathcal{N}$ such that $V \backslash W_0$ contains the intersection of open balls corresponding to $N_1, \ldots, N_l$ centered at v. Using this, one can verify that

$$|\lambda(z)| \le C \max_{1 \le j \le l} N_j(z) \tag{14.4}$$

for some $C \ge 0$ and every $z \in W$. Of course, this is trivial when $z \in W_0$, and otherwise one can use homogeneity to reduce to the case where $z = w + v$ for some $w \in W_0$, which is handled by the previous condition on $N_1, \ldots, N_l$. The Hahn–Banach theorem then implies that λ can be extended to a linear functional on V that satisfies the same condition, and which is therefore continuous with respect to the topology on V determined by $\mathcal{N}$, as desired.

15 Weak topologies

Let V be a vector space over the real or complex numbers. If λ is a linear functional on V, then it is easy to see that

$$N_\lambda(v) = |\lambda(v)| \tag{15.1}$$

is a seminorm on V. If Λ is a collection of linear functionals on V, then let $\mathcal{N}(\Lambda)$ be the corresponding collection of seminorms on V, consisting of N_λ with $\lambda \in \Lambda$. This is obviously a nice collection of seminorms on V when Λ is a nice collection of linear functionals on V, in the sense that for each $v \in V$ with $v \ne 0$ there is a $\lambda \in \Lambda$ such that $\lambda(v) \ne 0$. In this case, $\mathcal{N}(\Lambda)$ determines a topology on V such that V is a locally convex topological vector space, as before.

Of course, each $\lambda \in \Lambda$ is obviously continuous on V with respect to the topology determined by $\mathcal{N}(\Lambda)$. If μ is a linear functional on V which is a linear combination of finitely many elements of Λ, then it follows that μ is also continuous on V with respect to the topology determined by $\mathcal{N}(\Lambda)$. Conversely, if μ is a linear functional on V which is continuous with respect to the topology determined by $\mathcal{N}(\Lambda)$, then there are finitely many elements $\lambda_1, \ldots, \lambda_l$ of Λ such that

$$|\mu(v)| \le C \max_{1 \le j \le l} |\lambda_j(v)| \tag{15.2}$$

for some $C \ge 0$ and every $v \in V$. In particular, this implies that the kernel of μ contains the intersection of the kernels of $\lambda_1, \ldots, \lambda_l$, which implies in turn that μ can be expressed as a linear combination of $\lambda_1, \ldots, \lambda_l$. More precisely, this is a bit simpler when $\lambda_1, \ldots, \lambda_l$ are linearly independent as linear functionals on V, and it is easy to reduce to this case.

Suppose now that V is already a topological vector space, and let V' be the topological dual space of V, consisting of all continuous linear functionals on V. This is a vector space in a natural way, with respect to pointwise addition and scalar multiplication of linear functionals on V. Let us also ask that V' be a nice collection of linear functionals on V, so that for each $v \in V$ with $v \ne 0$ there is a continuous linear functional λ on V such that $\lambda(v) \neq 0$. This condition holds automatically when V is locally convex, as in the previous section. Although this condition fails for $L^p(X)$ when X is the unit interval equipped with Lebesgue measure and $0 < p < 1$, it does hold when X is any set equipped with counting measure for every $p > 0$.

If we take $\Lambda = V'$ in the previous discussion, then the topology determined by the corresponding collection $\mathcal{N}(V')$ of seminorms on V is known as the weak topology on V. It is easy to see that every open set in V with respect to the weak topology is also open with respect to the original topology on V, since the elements of V' are continuous on V with respect to the original topology by hypothesis. However, if the original topology on V is defined by a norm N, for instance, and if V is infinite-dimensional, then it is easy to see that open balls in V with respect to N are open sets with respect to the original topology on V and not with respect to the weak topology. In this case, every nonempty open subset of V with respect to the weak topology is unbounded with respect to the norm. By contrast, using the Hahn–Banach theorem, one can show that closed balls with respect to N are closed subsets of V with respect to the weak topology. If V is any locally convex topological vector space and W_0 is a closed linear subspace of V, then one can use the application of the Hahn–Banach theorem mentioned at the end of the previous section to show that W_0 is also closed with respect to the weak topology on V. Using a more precise version of the Hahn–Banach theorem, one can show that every closed convex subset of a locally convex topological vector space is closed with respect to the weak topology as well.

16 The weak* topology

Let W be a vector space over the real or complex numbers, and let W^* be the algebraic dual space of all linear functionals on W. Thus W^* is a vector space with respect to pointwise addition and scalar multiplication of linear functionals. If $w \in W$, then

$$\lambda \mapsto \lambda(w) \tag{16.1}$$

defines a linear functional on W, and hence

$$N_w^*(\lambda) = |\lambda(w)| \tag{16.2}$$

defines a seminorm on W^*. The collection of all of these seminorms N_w^* with $w \in W$ is automatically a nice collection of seminorms on W^*, because a linear functional λ on W is nonzero exactly when there is a $w \in W$ such that $\lambda(w) \neq 0$. The topology on W^* determined by this collection of seminorms is known as the weak* topology on W^*. Similarly, if V is a linear subspace of W^*, then the topology on V determined by the restrictions of these seminorms N_w^*, $w \in W$, to V is known as the weak* topology on V. This is the same as the topology induced on V by the weak* topology on W^*. In particular, if W is a topological vector space, then this can be applied to the topological dual W' of continuous linear functionals on W.

As an example, let X be a nonempty set, and let V be the vector space of real or complex-valued functions on X, with respect to pointwise addition and multiplication. If $f \in V$, then the support of f is defined by

$$\operatorname{supp} f = \{x \in X : f(x) \neq 0\}. \tag{16.3}$$

Let W be the linear subspace of V consisting of functions with finite support. If $g \in V$, then put

$$\lambda_g(f) = \sum_{x \in X} f(x)\, g(x) \tag{16.4}$$

for each $f \in W$. Although X may have infinitely many elements, the sum on the right side reduces to a finite sum, because only finitely many terms are different from 0. Thus the sum on the right makes sense, and defines a linear functional λ_g on W.

If λ is any linear functional on W, then λ can be represented as λ_g for some $g \in V$. More precisely, for each $y \in X$, let $\delta_y(x)$ be the function on X which is equal to 1 when $x = y$ and to 0 when $x \neq y$. If

$$g(y) = \lambda(\delta_y) \tag{16.5}$$

for each $y \in X$, then it is easy to see that

$$\lambda(f) = \lambda_g(f) \tag{16.6}$$

for every $f \in W$. This is because every $f \in W$ can be expressed as a linear combination of δ_y's for finitely many $y \in X$, and the equation holds by definition of $g(y)$ when $f = \delta_y$ for some $y \in X$. It is easy to see that the mapping from $g \in V$ to $\lambda_g \in W^*$ defines a one-to-one linear mapping from V onto W^*, so that V can be identified with W^*.

In this case, the weak* topology on V is the same as the topology described in Section 10. Remember that the latter topology was defined using the seminorms

$$N_y(g) = |g(y)| \tag{16.7}$$

on V with $y \in V$. If we identify V with W^* as in the preceding paragraph, then the weak* topology is defined by the seminorms

$$N_f^*(g) = \Big|\sum_{x \in X} f(x)\, g(x)\Big| \tag{16.8}$$

on V with $f \in W$. Of course, this reduces to (16.7) when $f = \delta_y$. Conversely,

$$N_f^*(g) \le \sum_{x \in X} |f(x)|\, |g(x)| = \sum_{x \in X} |f(x)|\, N_x(g) \tag{16.9}$$

for each $f \in W$ and $g \in V$, which implies that the seminorms $N_x(g)$ on V with $x \in X$ determine the same topology as the seminorms $N_f^*(g)$ with $f \in W$.

17 Weak* compactness

Let X be a nonempty set, let V be the vector space of real or complex-valued functions on X, and let W be the linear subspace of V consisting of functions on X with finite support. Thus we can identify V with the algebraic dual W^* of W, as in the previous section, in which case the weak* topology corresponds exactly to the one already defined in Section 10. As in Section 10, Tychonoff's theorem implies that closed and bounded subsets of V are compact with respect to this topology. Of course, this reduces to the fact that closed and bounded subsets of $\mathbf{R}^n$ or $\mathbf{C}^n$ are compact when X has only finitely many elements. If X has only finitely or countably many elements, then this topology on V is metrizable, and compactness is equivalent to sequential compactness.

Now let W be any vector space over the real or complex numbers. Suppose that A is a set, and that $\{w_\alpha\}_{\alpha \in A}$ is a family of vectors in W indexed by A. If $\{a_\alpha\}_{\alpha \in A}$ is a family of real or complex numbers, as appropriate, indexed by A and satisfying $a_\alpha = 0$ for all but finitely many $\alpha \in A$, then the sum

$$\sum_{\alpha \in A} a_\alpha \, w_\alpha \tag{17.1}$$

can be defined as an element of W. If every element of W can be expressed as a sum (17.1) with $a_\alpha = 0$ for all but finitely many $\alpha \in A$, then we say that $\{w_\alpha\}_{\alpha \in A}$ spans W. If (17.1) is equal to 0 only when $a_\alpha = 0$ for every $\alpha \in A$, then we say that $\{w_\alpha\}_{\alpha \in A}$ is linearly independent in W. Equivalently, $\{w_\alpha\}_{\alpha \in A}$ is linearly independent if the representation of any vector in W as a sum of the form (17.1) is unique. A family of vectors $\{w_\alpha\}_{\alpha \in A}$ of vectors in W is said to be a basis of W if it is linearly independent and spans W. This is the same as saying that every element of W can be represented in a unique way as a sum (17.1), where $a_\alpha = 0$ for all but finitely many $\alpha \in A$, as usual.

If W is the vector space of functions with finite support on a set X, then the functions δ_y, $y \in X$, defined in the previous section form a basis for W. Conversely, if $\{w_\alpha\}_{\alpha \in A}$ is a basis for a vector space W, then the mapping from $\{a_\alpha\}_{\alpha \in A}$ to (17.1) defines an isomorphism from the space of real or complex-valued functions with finite support on A, as appropriate, onto W. Well-known arguments based on the axiom of choice imply that every vector space W has a basis. It follows that closed and bounded subsets of the algebraic dual W^* of any real or complex vector space W are compact with respect to the weak* topology.

Now suppose that W is a topological vector space over the real or complex numbers. Let U be an open subset of W that contains 0, let C be a nonnegative real number, and put

$$B^*_{U,C} = \{\lambda \in W' : |\lambda(w)| \le C \text{ for every } w \in U\}, \tag{17.2}$$

where W' denotes the topological dual of W, as before. Equivalently,

$$B^*_{U,C} = \{\lambda \in W^* : |\lambda(w)| \le C \text{ for every } w \in U\}, \tag{17.3}$$

because any linear functional on W that is bounded on U is continuous. It is easy to see that $B^*_{U,C}$ is a closed and bounded subset of W^* with respect to the weak* topology, and hence is compact. This is the theorem of Banach and Alaoglu.

18 Uniform boundedness

Let M be a complete metric space, and let E be a collection of continuous real or complex-valued functions on M. Suppose that E is uniformly bounded pointwise on M, in the sense that

$$E(x) = \{f(x) : f \in E\} \tag{18.1}$$

is a bounded set of real or complex numbers for each $x \in M$. Put

$$A_n = \{x \in M : |f(x)| \le n \text{ for each } f \in E\} \tag{18.2}$$

for each positive integer n, so that

$$\bigcup_{n=1}^{\infty} A_n = M \tag{18.3}$$

by hypothesis. It is easy to see that A_n is a closed set in M for each n, because every $f \in E$ is continuous. If the interior of A_n is empty for each n, then the Baire category theorem would imply that the interior of $\bigcup_{n=1}^{\infty} A_n$ is also empty, a contradiction. Thus the interior of A_n is not empty for some positive integer n, which means that the functions $f \in E$ are uniformly bounded on a nonempty open subset of M. We would like to consider variants of this for linear functionals on topological vector spaces.

Let V be a topological vector space over the real or complex numbers. A sequence $\{v_j\}_{j=1}^{\infty}$ of vectors in V is said to be a *Cauchy sequence* if for each open set U in V with $0 \in U$ there is an $L \ge 0$ such that

$$v_j - v_l \in U \tag{18.4}$$

for every $j, l \ge L$. It is easy to check that every convergent sequence in V is a Cauchy sequence, using the continuity of addition on V at 0 in the usual way. Let us say that V is sequentially complete as a topological vector space if every Cauchy sequence of elements of V in this sense converges to an element of V.

If there is a countable local base for the topology of V at 0, then there is a translation-invariant metric on V that determines the same topology on V, as in Section 12. One can also check that a sequence of elements of V is a Cauchy sequence with respect to such a metric if and only if it is a Cauchy sequence in the previous sense for topological vector spaces. Thus V is complete as a metric space with respect to such a metric if and only if V is sequentially complete as a topological vector space. A topological vector space V is said to be an *F-space* if there is a countable local base for the topology of V at 0 and V is sequentially complete as a topological vector space. Equivalently, V is an F-space if it is complete as a metric space with respect to any translation-invariant metric that determines the same topology.

Let E be a collection of continuous linear functionals on V which is bounded with respect to the weak* topology on V', which is the same as saying that E is bounded pointwise on V. If V is an F-space, then the Baire category theorem holds on V, and the earlier argument implies that there is a nonempty open set U in V such that the elements of E are uniformly bounded on U. Without loss of generality, we may also ask that 0 be an element of U, since otherwise we can replace U with $U - u_0$ for some $u_0 \in U$. This is a version of the theorem of Banach and Steinhaus.

19 Totally bounded sets

Remember that a subset E of a metric space M is said to be totally bounded if for each $\epsilon > 0$, E can be covered by finitely many balls of radius ϵ. Now let V be a topological vector space over the real or complex numbers. A set $E \subseteq V$ is said to be *totally bounded* if for each open set U in V with $0 \in U$, E can be covered by finitely many translates of U. More precisely, this means that there are finitely many vectors $v_1, \ldots, v_n$ in V such that

$$E \subseteq \bigcup_{j=1}^{n} (v_j + U), \tag{19.1}$$

or equivalently that there is a finite set $A \subseteq V$ such that

$$E \subseteq A + U. \tag{19.2}$$

Thus finite sets are automatically totally bounded. Of course, any set $E \subseteq V$ can be covered by a collection of translates of an open set $U \subseteq V$ that contains 0, using all translates $v + U$ with $v \in E$. If E is compact, then such a covering can be reduced to a finite subcovering, so that E is totally bounded, as in the case of metric spaces.

If there is a countable local base for the topology of V at 0, then there is a translation-invariant metric on V that determines the same topology on V, as in Section 12. If $d(v, w)$ is such a metric, then it is easy to see that $E \subseteq V$ is totally bounded as a subset of V as a topological vector space if and only if E is totally bounded as a subset of V as a metric space with respect to $d(v, w)$.

If E_1, E_2 are totally bounded subsets of V, then $E_1 \cup E_2$ is also totally bounded, because one can simply combine coverings of E_1 and E_2 by finitely many translates of an open set $U \subseteq V$ that contains 0 to cover $E_1 \cup E_2$ by finitely many translates of U. To check that $E_1 + E_2$ is totally bounded, let U be any open set in V that contains 0, and let U_1, U_2 be open subsets of V that contain 0 and satisfy $U_1 + U_2 \subseteq U$. Because E_1, E_2 are totally bounded, there are finite sets $A_1, A_2 \subseteq V$ such that

$$E_1 \subseteq A_1 + U_1, \quad E_2 \subseteq A_2 + U_2. \tag{19.3}$$

This implies that

$$E_1 + E_2 \subseteq (A_1 + A_2) + (U_1 + U_2) \subseteq (A_1 + A_2) + U, \tag{19.4}$$

and hence that $E_1 + E_2$ is totally bounded, since $A_1 + A_2$ is finite.

Subsets of totally bounded sets are obviously totally bounded too. Suppose that $E \subseteq V$ is totally bounded, and let us check that the closure $\overline{E}$ of E is totally bounded as well. Let U be an open set in V that contains 0, and let W be an open set in V that contains 0 and satisfies $\overline{W} \subseteq U$, which exists because V is regular as a topological space. Thus E can be covered by finitely many translates of W, because E is totally bounded. This implies that $\overline{E}$ can be covered by finitely many translates of $\overline{W}$, and hence by finitely many translates of U, as desired.

Let us check that $E \subseteq V$ is bounded when E is totally bounded. Let U be an open set in V that contains 0, and let U_1, U_2 be open subsets of V that contain 0 and satisfy $U_1 + U_2 \subseteq U$. It will be helpful to ask also that U_2 be balanced, which can always be arranged by replacing U_2 with a smaller neighborhood of 0, if necessary. If $E \subseteq V$ is totally bounded, then $E \subseteq A + U_2$ for some finite set $A \subseteq V$. Let r be a nonnegative real number such that $A \subseteq t\, U_1$ when $|t| \geq r$, which exists by the absorbing property of open subsets of V that contain 0. Thus

$$E \subseteq A + U_2 \subseteq t\, U_1 + U_2 \tag{19.5}$$

when $|t| \geq r$. If $|t| \geq 1$, then $U_2 \subseteq t\, U_2$, because U_2 is balanced, and hence

$$t\, U_1 + U_2 \subseteq t\, U_1 + t\, U_2 \subseteq t\, U. \tag{19.6}$$

This shows that $E \subseteq t\, U$ when $|t| \geq \max(r, 1)$, as desired.

If E is a totally bounded subset of a locally convex topological vector space V, then the convex hull $\mathrm{con}(E)$ is also totally bounded. To see this, let U be an open set in V that contains 0, and let U_1 and U_2 be open subsets of V that contain 0 and satisfy $U_1 + U_2 \subseteq U$. Because V is locally convex, we may also ask that U_2 be convex, since otherwise we can replace U_2 with a convex open subset that also contains 0. Using the hypothesis that E be totally bounded, we get a finite set $A \subseteq V$ such that $E \subseteq A + U_2$. Observe that

$$\mathrm{con}(E) \subseteq \mathrm{con}(A) + \mathrm{con}(U_2) = \mathrm{con}(A) + U_2, \tag{19.7}$$

where the second step follows from the convexity of U_2. To get the first step, remember that $\mathrm{con}(E)$ consists of convex combinations of elements of E, and that each element of E is a sum of elements of A and U_2. By rearranging the sums, one can express any element of $\mathrm{con}(E)$ as a sum of convex combinations of elements of A and of U_2, as desired.

If $a_1, \ldots, a_n$ is a list of the elements of A, then every element of $\mathrm{con}(A)$ can be expressed as

$$\sum_{j=1}^{n} t_j\, a_j, \tag{19.8}$$

where $t_1, \ldots, t_n$ are nonnegative real numbers such that $\sum_{j=1}^n t_j = 1$. Of course,

$$\Big\{t \in \mathbf{R}^n : t_j \ge 0 \text{ for each } j = 1, \ldots, n \text{ and } \sum_{j=1}^n t_j = 1\Big\} \tag{19.9}$$

is a compact subset of $\mathbf{R}^n$, which implies that $\mathrm{con}(A)$ is a compact subset of V, since it is the image of (19.9) under a continuous mapping from $\mathbf{R}^n$ into V. In particular, $\mathrm{con}(A)$ is totally bounded, so that

$$\mathrm{con}(A) \subseteq B + U_1 \tag{19.10}$$

for some finite set $B \subseteq V$. It follows that

$$\mathrm{con}(E) \subseteq \mathrm{con}(A) + U_2 \subseteq B + U_1 + U_2 \subseteq B + U, \tag{19.11}$$

as desired.

20 Another example

Let X be a nonempty set, and let V be the vector space of real or complex-valued functions on X. If ρ is a positive real-valued function on X and $f \in V$, then put

$$B_\rho(f) = \{g \in V : |f(x) - g(x)| < \rho(x) \text{ for every } x \in X\}. \tag{20.1}$$

Let us say that $U \subseteq V$ is an open set if for each $f \in U$ there is a positive real-valued function ρ on X such that

$$B_\rho(f) \subseteq U. \tag{20.2}$$

It is easy to see that this defines a topology on V, and that $B_\rho(f)$ is an open set in V with respect to this topology for each positive real-valued function ρ on X and $f \in V$. If we identify V with the Cartesian product of copies of $\mathbf{R}$ or $\mathbf{C}$ indexed by elements of X, then this topology on V corresponds exactly to the "strong product topology" generated by arbitary products of open subsets of the factors.

Let us suppose from now on that X has infinitely many elements, since otherwise this would be the same as the topology discussed in Section 10, and V would be isomorphic to $\mathbf{R}^n$ or $\mathbf{C}^n$ with the standard topology for some n. It is easy to see that pointwise addition of functions is continuous with respect to this topology on V, as is the mapping $f \mapsto t\, f$ for a fixed real or complex number t, as appropriate. However, if $f \in V$ satisfies $f(x) \neq 0$ for infinitely many $x \in X$, then one can check that $t \mapsto t\, f$ is not continuous as a mapping from $\mathbf{R}$ or $\mathbf{C}$ into V, as appropriate. Thus V is not a topological vector space with respect to this topology. Note that V is Hausdorff and even regular with respect to this topology.

If W is the linear subspace of V consisting of functions on X with finite support, then it is easy to see that W is a locally convex topological vector space with respect to the topology induced by the one just defined on V. Equivalently, if ρ is a positive real-valued function on X, then

$$N_\rho(f) = \max_{x \in X} \rho(x)^{-1} \, |f(x)| \tag{20.3}$$

defines a norm on W, and the topology on W determined by the collection of all of these norms is the same as the one induced by the topology on V described in the previous paragraph. More precisely, the open ball in W centered at $f \in W$ with radius 1 with respect to N_ρ is the same as the intersection of the set $B_\rho(f)$ defined before with W. Note that W is also a closed subset of V with respect to the topology defined in the previous paragraph.

Let $\rho_1, \rho_2, \ldots$ be a sequence of positive real-valued functions on X, and let $x_1, x_2, \ldots$ be a sequence of distinct elements of X. Also let ρ be another positive real-valued function on X such that

$$\rho(x_n) = n^{-1} \min_{1 \le j \le n} \rho_j(x_n) \tag{20.4}$$

for each $n \geq 1$. In this case, one can check that N_ρ is not bounded by a constant multiple of N_{ρ_n} on W for any n, and hence that no sequence of these norms is sufficient to determine this topology on W. Equivalently, there is no countable local base for the topology at 0, and thus W is not metrizable, even when X has only countably many elements.

Let us define the support of a set $E \subseteq W$ by

$$\operatorname{supp} E = \{x \in X : f(x) \neq 0 \text{ for some } f \in E\}, \tag{20.5}$$

which is the same as the union of the supports of the elements of E. Suppose that the support of E is infinite, and let $\{x_n\}_{n=1}^\infty$ be a sequence of distinct elements of $\operatorname{supp} E$. Also let $\{f_n\}_{n=1}^\infty$ be a sequence of elements of E such that $f_n(x_n) \neq 0$ for each n. If ρ is a positive real-valued function on X such that

$$\rho(x_n) = n^{-1} \, |f_n(x_n)| \tag{20.6}$$

for each n, then

$$N_\rho(f_n) \geq \rho(x_n) \, |f(x_n)| = n \tag{20.7}$$

for each n, so that N_ρ is not bounded on E. Thus E is not bounded in W when the support of E is finite.

Equivalently, the support of a bounded set $E \subseteq W$ is finite. If the support of E is finite, then the boundedness of E in W reduces to the boundedness of $|f(x)|$ for $f \in E$ and each $x \in \operatorname{supp} E$. In particular, closed and bounded sets in W are compact, because of the classical results for $\mathbf{R}^n$ and $\mathbf{C}^n$.

In the next section, we shall consider another collection of norms on W, which is equivalent to this one when X is countable.

21 Variations

Let X be an infinite set, and let W be the vector space of real or complex-valued functions on X with finite support, as in the preceding section. If ρ is a positive real-valued function on X, then

$$\widetilde{N}_\rho(f) = \sum_{x \in X} \rho(x)^{-1} \, |f(x)| \tag{21.1}$$

defines a norm on W. More precisely, this sum reduces to a finite sum when $f \in W$, and thus makes sense. If $N_\rho(f)$ is as in (20.3), then

$$N_\rho(f) \le \widetilde{N}_\rho(f) \tag{21.2}$$

for every $f \in W$. This implies that the topology determined on W by the collection of all of the norms $\widetilde{N}_\rho$ is at least as strong as the topology determined by the norms N_ρ, in the sense that every open subset of W with respect to the topology defined by the norms N_ρ is also an open set with respect to the topology defined by the norms $\widetilde{N}_\rho$.

If X is countably infinite, then there is a positive real-valued function a on X such that

$$\sum_{x \in X} a(x) = 1. \tag{21.3}$$

More precisely, this sum may be defined as the supremum of the sum of $a(x)$ over any finite set of $x \in X$. Equivalently, if $\{x_j\}_{j=1}^\infty$ is a sequence of elements of X in which every element of X occurs exactly once, then we ask that

$$\sum_{j=1}^\infty a(x_j) = 1, \tag{21.4}$$

which implicitly includes the convergence of this infinite series. At any rate, if ρ is a positive real-valued function on X and $f \in W$, then

$$\begin{aligned} \widetilde{N}_\rho(f) = \sum_{x \in X} \rho(x)^{-1} \, |f(x)| &= \sum_{x \in X} a(x) \, a(x)^{-1} \, \rho(x)^{-1} \, |f(x)| \\ &\le N_{a\rho}(f) \sum_{x \in X} a(x) = N_{a\rho}(f). \end{aligned} \tag{21.5}$$

This implies that the topology on W determined by the collection of norms of the form $\widetilde{N}_\rho$ is the same as the topology on W determined by the collection of norms of the form N_ρ when X is countable.

If $y \in X$, then let $\delta_y(x)$ be the function on X equal to 1 when $x = y$ and to 0 when $x \ne y$. If N is any seminorm on W, then it follows that

$$N(f) \le \sum_{y \in X} N(\delta_y) \, |f(y)| \tag{21.6}$$

for each $f \in W$. To see this, one can express f as a linear combination of δ_y's,

$$f(x) = \sum_{y \in \operatorname{supp} f} f(y)\, \delta_y(x), \tag{21.7}$$

and then use the hypothesis that N is a seminorm on W. If ρ is a positive real-valued function on X such that $N(\delta_y) \le \rho(y)^{-1}$ for each $y \in X$, then we get that

$$N(f) \le \widetilde{N}_\rho(f) \tag{21.8}$$

for every $f \in W$. It follows that the topology on W determined by the collection of norms of the form $\widetilde{N}_\rho$ is the same as the topology determined by the collection of all seminorms on W.

In particular, every linear functional on W is continuous with respect to the topology determined by the collection of norms $\widetilde{N}_\rho$. If X is countable, then it follows that every linear functional on W is continuous with respect to the topology determined by the collection of norms of the form N_ρ, as in the previous section.

22 Continuous functions

Let X be a locally compact Hausdorff topological space, and let $C(X)$ be the vector space of real or complex-valued continuous functions on X, with respect to pointwise addition and scalar multiplication. If K is a nonempty compact subset of X, then put

$$N_K(f) = \sup_{x \in K} |f(x)| \tag{22.1}$$

for each $f \in C(X)$. It is easy to see that this defines a seminorm on $C(X)$, known as the supremum seminorm associated to K. The collection of all of these seminorms is a nice collection of seminorms on $C(X)$, as in Section 9, and thus defines a topology on $C(X)$ which makes $C(X)$ into a locally convex topological vector space. If X is equipped with the discrete topology, so that $C(X)$ consists of all real or complex-valued functions on X, then this topology is the same as the one considered in Section 10.

If X is compact, then we can take $K = X$ in (22.1), to get a single norm on $C(X)$ which determines the same topology. Otherwise, suppose that X is σ-compact, which means that X can be expressed as the union of a sequence of compact subsets. This holds for instance when X is a countable set, or an open set in $\mathbf{R}^n$ for some n. In this case, one can combine σ-compactness with local compactness to get a sequence $K_1, K_2, \ldots$ of compact subsets of X such that K_l is contained in the interior of K_{l+1} for each l and $\bigcup_{l=1}^\infty K_l = X$. If K is any compact subset of X, then $K \subseteq K_l$ for some l, because K is covered by the union of the interiors of the K_l's by construction, and hence K is contained in the union of the interiors of finitely many K_l's by compactness. This implies that

$$N_K(f) \le N_{K_l}(f) \tag{22.2}$$

for each $f \in C(X)$, so that the supremum seminorms associated to the K_l's are sufficient to determine the same topology on $C(X)$. It follows that this topology on $C(X)$ is metrizable when X is σ-compact, as in Section 12.

Remember that a sequence $\{v_j\}_{j=1}^\infty$ of vectors in a topological space V is said to be a Cauchy sequence if for each open set $U \subseteq V$ with $0 \in U$ we have that $v_j - v_l \in U$ for all sufficiently large j, l. If $V = C(X)$ with the topology just defined, then this is equivalent to saying that a sequence of continuous functions $\{f_j\}_{j=1}^\infty$ on X is a Cauchy sequence if for each nonempty compact set $K \subseteq X$ and every $\epsilon > 0$ there is an $L \geq 1$ such that

$$N_K(f_j - f_l) < \epsilon \tag{22.3}$$

for every $j, l \geq L$. In particular, this implies that $\{f_j(x)\}_{j=1}^\infty$ is a Cauchy sequence in $\mathbf{R}$ or $\mathbf{C}$, as appropriate, for each $x \in X$. The completeness of the real or complex numbers implies that $\{f_j(x)\}_{j=1}^\infty$ converges to a real or complex number $f(x)$ for each $x \in X$. Using the uniform version of the Cauchy condition (22.3), one can check that $\{f_j\}_{j=1}^\infty$ converges to f uniformly on compact subsets of X under these conditions. This implies that $\{f_j\}_{j=1}^\infty$ converges to f uniformly on a neighborhood of any element p of X, and hence that f is continuous at p. Thus f is also a continuous function on X, and $\{f_j\}_{j=1}^\infty$ converges to f in the topology of $C(X)$.

23 Nets

A partially-ordered set $(A, \prec)$ is said to be a *directed system* if for every $a, b \in A$ there is a $c \in A$ such that $a \prec c$ and $b \prec c$. Of course, if A is linearly ordered, so that $a \prec b$ or $b \prec c$, then one can simply take c to be a or b. A net of elements of a set X indexed by A may be denoted $\{x_a\}_{a \in A}$, and is really just a function defined on A with values in X. In particular, a sequence is the same as a net indexed by the set $\mathbf{Z}_+$ of positive integers, with the standard ordering. The ordering on the set of indices of a net or sequence is important for convergence conditions on the net or sequence, as follows.

Suppose that X is a topological space, and that x is an element of X. A net $\{x_a\}_{a \in A}$ of elements of X is said to converge to x in X if for each open set U in X with $x \in U$ there is an $a \in A$ such that $x_b \in U$ for every $b \in A$ that satisfies $a \prec b$. This reduces to the usual notion of convergence of a sequence when A is the set of positive integers with the standard ordering. It is easy to see that the limit of a convergent net of elements of X is unique when X is Hausdorff. Conversely, if X is not Hausdorff, then there is a net of elements of X with a suitable set of indices that converges to more than one element of X.

Let p be an element of X, and let $\mathcal{B}(p)$ be a local base for the topology of X at p. This means that $\mathcal{B}(p)$ is a collection of open subsets of X that each contain p as an element, and that for each open set W in X with $p \in W$ there is a $U \in \mathcal{B}(p)$ such that $U \subseteq W$. Consider the ordering $\prec$ on $\mathcal{B}(p)$ where $U_1 \prec U_2$ for $U_1, U_2 \in \mathcal{B}(p)$ when $U_2 \subseteq U_1$. This is clearly a partial ordering on $\mathcal{B}(p)$, and to check that $\mathcal{B}(p)$ is a directed system with respect to $\prec$, let $U_1, U_2 \in \mathcal{B}(p)$ be given. Thus $U_1 \cap U_2$ is an open set in X that contains p as an element, so that there is an element U_3 of $\mathcal{B}(p)$ such that $U_3 \subseteq U_1 \cap U_2$, because $\mathcal{B}(p)$ is a local base for the topology of X at p. Equivalently, $U_3 \subseteq U_1$ and $U_3 \subseteq U_2$, which is the same as saying that $U_1 \prec U_3$ and $U_2 \prec U_3$, as desired. Of course, the collection of all open subsets of X that contain p is a local base for the topology of X at p.

If f is a mapping from X into another topological space Y which is continuous at $p \in X$, and if $\{x_a\}_{a \in A}$ is a net of elements of X that converges to p in X, then it is easy to see that $\{f(x_a)\}_{a \in A}$ converges to $f(p)$ as a net of elements of Y. Conversely, if f is not continuous at p, then there is a net $\{x_a\}_{a \in A}$ of elements of X that converges to p in X such that $\{f(x_a)\}_{a \in A}$ does not converge to $f(p)$ in Y. More precisely, if f is not continuous at p, then there is an open set W in Y such that $f(p) \in W$ but $f(U) \not\subseteq W$ for every open set $U \subseteq X$ with $p \in U$. Let $\mathcal{B}(p)$ be a local base for the topology of X at p, and for each $U \in \mathcal{B}(p)$, let $x(U)$ be an element of U such that $f(x(U))$ is not an element of W. It is easy to see that the $x(U)$'s converge to p as a net of elements of X indexed by $\mathcal{B}(p)$ as in the previous paragraph, while the $f(x(U))$'s do not converge to $f(p)$ in Y, because $f(x(U)) \notin W$ for each $U \in \mathcal{B}(p)$.

Similarly, if a net $\{x_a\}_{a \in A}$ of elements of a set $E \subseteq X$ converges to a point $p \in X$, then it is easy to see that p is an element of the closure $\overline{E}$ of E in X. Conversely, if $p \in \overline{E}$, then for each open set $U \subseteq X$ with $p \in U$ there is an element $x(U)$ of U which is also an element of E. If $\mathcal{B}(p)$ is a local base for the topology of X at p, then the $x(U)$'s with $U \in \mathcal{B}(p)$ form a net of elements of E that converges to p in X, with respect to the ordering of $\mathcal{B}(p)$ be reverse inclusion, as before.

Suppose that $U_1, U_2, \ldots$ is a sequence of open subsets of X, each of which contains a point $p \in X$, and such that the collection of U_j's forms a local base for the topology of X at p. Without loss of generality, we may also ask that $U_{j+1} \subseteq U_j$ for each j, since otherwise we can replace each U_j with the intersection of $U_1, \ldots, U_j$. Thus the collection of U_j's is linearly ordered by inclusion under these conditions, and it is easy to see that we can get sequences converging to p in the context of the previous two paragraphs.

In some circumstances, one can define a Cauchy condition for nets, in analogy with Cauchy sequences. We shall continue with this in the next section.

24 Cauchy nets

A net $\{x_a\}_{a \in A}$ of elements of a metric space $(M, d(x,y))$ is said to be a Cauchy net if for each $\epsilon > 0$ there is an $a \in A$ such that

$$d(x_b, x_c) < \epsilon \tag{24.1}$$

for every $b, c \in A$ that satisfy $a \prec b$ and $a \prec c$. This clearly reduces to the usual definition of a Cauchy sequence when $A = \mathbf{Z}_+$ with the standard ordering. Conversely, if $\{x_a\}_{a \in A}$ is a Cauchy net in M, then for each positive integer j

there is an $a_j \in A$ such that

$$d(x_b, x_c) < 1/j \tag{24.2}$$

for every $b, c \in A$ with $a_j \prec b, c$. One can also choose the a_j's recursively so that $a_j \prec a_{j+1}$ for each j as well, although we do not really need this here. At any rate, it is east to see that $\{x_{a_j}\}_{j=1}^\infty$ is a Cauchy sequence in M. If $\{x_{a_j}\}_{j=1}^\infty$ converges to an element x of M, then one can also check that $\{x_a\}_{a \in A}$ converges to x as a net of elements of M. In particular, if M is complete as a metric space, then every Cauchy net of elements of M converges to an element of M. Of course, a convergent net of elements of any metric space is a Cauchy net, by the usual argument with the triangle inequality.

Now let V be a topological vector space over the real or complex numbers. A net $\{v_a\}_{a \in A}$ of elements of V is said to be a Cauchy net if for each open set U in V with $0 \in U$ there is an $a \in A$ such that

$$v_b - v_c \in U \tag{24.3}$$

for every $b, c \in A$ with $a \prec b, c$. This reduces to the notion of a Cauchy sequence in a topological vector space defined earlier when $A = \mathbf{Z}_+$ with the standard ordering. This also reduces to the preceding definition of a Cauchy net in a metric space when V is equipped with a translation-invariant metric that determines the same topology.

Suppose that $\{v_a\}_{a \in V}$ is a net of elements of V that converges to an element v of V, and let us check that $\{v_a\}_{a \in A}$ is a Cauchy net in V. Let U be an open set in V that contains 0, and let U_1, U_2 be open subsets of V that contain 0 and satisfy $U_1 + U_2 \subseteq U$, which exist because of the continuity of addition on V at 0, as usual. If $W = U_1 \cap (-U_2)$, then W is also an open set in V that contains 0, by the continuity of the mapping $w \mapsto -w$ on V, and

$$W + (-W) \subseteq U_1 + U_2 \subseteq U. \tag{24.4}$$

Of course, $v + W$ is an open set in V that contains v, and so the convergence of $\{v_a\}_{a \in A}$ to v in V implies that there is an $a \in A$ such that

$$v_b \in v + W \tag{24.5}$$

for every $b \in A$ with $a \prec b$. Equivalently,

$$v_b - v \in W \tag{24.6}$$

when $a \prec b$, and so

$$v_b - v_c = (v_b - v) - (v_c - v) \in W - W \subseteq U \tag{24.7}$$

for every $b, c \in A$ with $a \prec b, c$, as desired.

If there is a countable local base for the topology of V at 0, then there is a translation-invariant metric $d(v, w)$ on V that determines the same topology, as in Section 12. In this case, a Cauchy net $\{v_a\}_{a \in A}$ of elements of V as a topological vector space is the same as a Cauchy net of elements of V as a metric space with respect to $d(\cdot, \cdot)$, as mentioned earlier. The discussion of metric spaces at the beginnning of the section shows that we can extract a Cauchy sequence in V from this Cauchy net, whose convergence implies the convergence of the whole net. If every Cauchy sequence in V converges, then it follows that every Cauchy net in V converges, when there is a countable local base for the topology of V at 0. The proof of this could also be given more directly in terms of topological vector spaces, without using metrics explicitly.

25 Completeness

Let us say that a topological vector space V over the real or complex numbers is complete if every Cauchy net of elements of V converges in V. If there is a countable local base for the topology of V at 0, then this reduces to sequential completeness, as in the previous section. In particular, the real and complex numbers are complete as one-dimensional topological vector spaces, with their standard topologies, since they are complete as metric spaces with their standard metrics. Similarly, $\mathbf{R}^n$ and $\mathbf{C}^n$ are complete with respect to their standard topologies, for each positive integer n.

Let V be the vector space of real or complex-valued functions on a nonempty set X, equipped with the topology discussed in Section 10. If $\{f_a\}_{a \in A}$ is a Cauchy net of functions on X with respect to this topology, then it is easy to see that $\{f_a(x)\}_{a \in A}$ is a Cauchy net in $\mathbf{R}$ or $\mathbf{C}$, as appropriate, for each $x \in X$. Hence $\{f_a(x)\}_{a \in A}$ converges in $\mathbf{R}$ or $\mathbf{C}$, as appropriate, for each $x \in X$, by the completeness of $\mathbf{R}$ and $\mathbf{C}$. If $f(x)$ denotes the limit of $\{f_a(x)\}_{a \in A}$ for each $x \in X$, then $f(x)$ determines an element of V, and it is easy to see that $\{f_a\}_{a \in A}$ converges to f with respect to this topology on V. Thus V is complete with respect to this topology.

Now let V be the vector space of real or complex-valued functions on an infinite set X with the stronger topology discussed in Section 20. Although V is no longer a topological vector space, it is still a commutative topological group with respect to addition, and the same basic notions make sense as before. If $\{f_a\}_{a \in A}$ is a Cauchy net of functions on X with respect to this stronger topology, then it is easy to see that $\{f_a(x)\}_{a \in A}$ is again a Cauchy net in $\mathbf{R}$ or $\mathbf{C}$ for each $x \in X$, which therefore converges to a real or complex number $f(x)$ for each $x \in X$, as appropriate. To check that $\{f_a\}_{a \in A}$ converges to f with respect to this stronger topology on V, let ρ be a positive real-valued function on X. As in Section 20,

$$B_{\rho/2}(0) = \{g \in V : |g(x)| < \rho(x)/2 \text{ for every } x \in X\} \tag{25.1}$$

is an open subset of V that contains 0. Because $\{f_a\}_{a \in A}$ is supposed to be a Cauchy net in V, there is an $a \in A$ such that

$$f_b - f_c \in B_{\rho/2}(0) \tag{25.2}$$

for every $b, c \in A$ with $a \prec b, c$, which is to say that

$$|f_b(x) - f_c(x)| < \rho(x)/2 \tag{25.3}$$

for every $b, c \in A$ with $a \prec b, c$ and every $x \in X$. This implies that

$$|f_b(x) - f(x)| \le \rho(x)/2 < \rho(x) \tag{25.4}$$

for every $b \in A$ with $a \prec b$ and every $x \in X$, by taking the pointwise limit over c in (25.3). This shows that $\{f_a\}_{a \in A}$ converges to f with respect to the strong topology on V, as desired.

We also saw in Section 20 that the linear subspace W of V consisting of functions with finite support on X is a topological vector space with respect to the topology induced by the one on V. It is easy to see that W is complete as a topological vector space, using the remarks in the preceding paragraph, and the fact that W as a subset of V. More precisely, if $\{f_a\}_{a \in A}$ is a Cauchy net in W, then $\{f_a\}_{a \in A}$ converges to some $f \in V$ in the topology of V, as in the preceding paragraph. The fact that W is closed in V implies that $f \in W$, and hence that $\{f_a\}_{a \in A}$ converges in W, as desired.

Suppose that W is any vector space over the real or complex numbers, and that the algebraic dual W^* of W is equipped with the weak* topology, as in Section 16. If $\{\lambda_a\}_{a \in A}$ is a Cauchy net in W^*, then it is easy to see that $\{\lambda_a(w)\}_{a \in A}$ is a Cauchy net in $\mathbf{R}$ or $\mathbf{C}$, as appropriate, for each $w \in W$. This implies that $\{\lambda_a(w)\}_{a \in A}$ converges to a real or complex number $\lambda(w)$ for each $w \in W$, by the completeness of $\mathbf{R}$ and $\mathbf{C}$. One can also check that $\lambda(w)$ defines a linear functional on W, using the linearity of $\lambda_a(w)$ for each $a \in A$. It is easy to see that $\{\lambda_a\}_{a \in A}$ converges to λ with respect to the weak* topology on W^*, and hence that W^* is complete with respect to the weak* topology.

If X is a nonempty set, V is the vector space of real or complex-valued functions on X, and W is the linear subspace of V consisting of functions with finite support, then we can identify V with the algebraic dual W^* of W, as in Section 16. Conversely, any real or complex vector space W can be identified with the space of real or complex-valued functions with finite support on some set, using a basis as in Section 17. Thus this last example is basically the same as the one in Section 10.

Suppose that X is a locally compact Hausdorff topological space, and let $C(X)$ be the vector space of real or complex-valued continuous functions on X, with the topology discussed in Section 22. If $\{f_a\}_{a \in A}$ is a Cauchy net in $C(X)$, then $\{f_a(x)\}_{a \in A}$ is a Cauchy net in $\mathbf{R}$ or $\mathbf{C}$, as appropriate, for each $x \in X$, and hence converges to a real or complex number $f(x)$ for each $x \in X$. If K is a nonempty compact subset of X and $\epsilon > 0$, then there is an $a \in A$ such that

$$N_K(f_b - f_c) = \sup_{x \in K} |f_b(x) - f_c(x)| < \epsilon/2 \tag{25.5}$$

for every $b, c \in A$ with $a \prec b, c$, because $\{f_a\}_{a \in A}$ is a Cauchy net in $C(X)$. Thus

$$|f_b(x) - f_c(x)| < \epsilon/2 \tag{25.6}$$

for every $b, c \in A$ with $a \prec b, c$ and every $x \in K$, which implies that

$$|f_b(x) - f(x)| \le \epsilon/2 \tag{25.7}$$

for every $b \in A$ with $a \prec b$ and every $x \in K$, by passing to the limit in c. This shows that f can be approximated uniformly by continuous functions on each compact set, and hence that the restriction of f to any compact set in X is continuous. It follows that f is continuous on X, because X is locally compact. The previous remarks also imply that $\{f_a\}_{a \in A}$ converges to f in $C(X)$, so that $C(X)$ is complete.

Let V be any topological vector space over the real or complex numbers, and let W be a linear subspace of V. If $v \in V$ is an element of the closure $\overline{W}$ of W in V, then there is a net $\{w_a\}_{a \in A}$ of elements of W that converges to v. In particular, $\{w_a\}_{a \in A}$ is a Cauchy net in W, as a topological vector space with respect to the topology induced by the one on V. If W is complete as a topological vector space, then $\{w_a\}_{a \in A}$ converges to an element of W, which is equal to v by the uniqueness of limits of convergent nets in Hausdorff topological spaces. This implies that W is a closed linear subspace of V when it is complete with respect to the topology induced by the one on V.

26 Filters

A *filter* on a set X is a nonempty collection $\mathcal{F}$ of nonempty subsets of X that satisfy the following two additional conditions. First, if A and B are elements of $\mathcal{F}$, then their intersection $A \cap B$ is also an element of $\mathcal{F}$. Second, if A is an element of $\mathcal{F}$ and E is a subset of X such that $A \subseteq E$, then E is an element of $\mathcal{F}$ as well.

If X is a topological space, then we say that a filter $\mathcal{F}$ on X converges to a point $p \in X$ if for each open set U in X with $p \in U$ we have that $U \in \mathcal{F}$. It is easy to see that the limit of a convergent filter on X is unique when X is Hausdorff. Conversely, if X is not Hausdorff, then one can show that there are filters on X that converge to more than one element of X.

A nonempty collection $\mathcal{F}_0$ of nonempty subsets of a set X is said to be a *pre-filter* if for any $A, B \in \mathcal{F}_0$ there is a $C \in \mathcal{F}_0$ such that

$$C \subseteq A \cap B. \tag{26.1}$$

In this case, it is easy to check that

$$\mathcal{F} = \{E \subseteq X : A \subseteq E \text{ for some } A \in \mathcal{F}_0\} \tag{26.2}$$

is a filter on X, and we say that $\mathcal{F}$ is generated by $\mathcal{F}_0$. Alternatively, if $\mathcal{F}$ is any filter on a set X, then we say that $\mathcal{B} \subseteq \mathcal{F}$ is a *base* for $\mathcal{F}$ if for each $E \in \mathcal{F}$ there is an $A \in \mathcal{B}$ such that $A \subseteq E$. Observe that a base $\mathcal{B}$ for a filter $\mathcal{F}$ on X is a pre-filter on X, and that $\mathcal{F}$ is generated by $\mathcal{B}$ as before. Similarly, a pre-filter $\mathcal{F}_0$ on X is a base for the filter $\mathcal{F}$ that it generates.

Suppose that $\mathcal{F}$ is a filter on a topological space X that converges to a point $p \in X$, and that E is a subset of X that is also an element of $\mathcal{F}$. If U is any open set in X that contains p as an element, then $U \in \mathcal{F}$, and hence $E \cap U \in \mathcal{F}$. This implies that $E \cap U \neq \emptyset$, so that p is an element of the closure $\overline{E}$ of E. Conversely, let p be an element of the closure of E, and let $\mathcal{F}_0$ be the collection of subsets of X of the form $E \cap U$, where U is an open subset of X that contains p as an element. Under these conditions, one can check that $\mathcal{F}_0$ is a pre-filter on X, and that the filter generated by $\mathcal{F}$ converges to p.

Let $(A, \prec)$ be a nonempty directed system, and let $\{x_a\}_{a \in A}$ be a net of elements of a set X indexed by A. Put

$$E_a = \{x_t : t \in A, a \prec t\} \tag{26.3}$$

for each $a \in A$, so that

$$E_b \subseteq E_a \tag{26.4}$$

when $a, b \in A$ and $a \prec b$. If a, b are any two elements of A, then there is another element c of A such that $a, b \prec c$, because $(A, \prec)$ is a directed system. If $\mathcal{F}_0$ is the collection of subsets of X of the form E_a with $a \in A$, then it follows that $\mathcal{F}_0$ is a pre-filter on X. If X is a topological space, then one can check that $\{x_a\}_{a \in A}$ converges to an element x of X if and only if the filter $\mathcal{F}$ generated by $\mathcal{F}_0$ converges to x.

Now let $\mathcal{F}$ be a filter on a set X, and let $\mathcal{B}$ be a base for $\mathcal{F}$. Of course, one can always take $\mathcal{B} = \mathcal{F}$, but in many situations there may be simpler choices. Let $\prec$ be the partial ordering defined on $\mathcal{B}$ by saying that $A \prec B$ when $A, B \in \mathcal{B}$ satisfy $B \subseteq A$. The fact that $\mathcal{B}$ is a pre-filter implies that $\mathcal{B}$ is a directed system with respect to this ordering.

Suppose that X is a topological space, and that $\mathcal{F}$ converges to a point $x \in X$. If $x(A)$ is an element of A for each $A \in \mathcal{B}$, then $\{x(A)\}_{A \in \mathcal{B}}$ converges as a net indexed by $\mathcal{B}$ to x in X, using the ordering on $\mathcal{B}$ described in the previous paragraph. Conversely, if $\mathcal{F}$ does not converge to x, then there is an open set U in X such that $x \in U$ and $U \notin \mathcal{F}$. This implies that $A \not\subseteq U$ for every $A \in \mathcal{F}$, by the definition of a filter, and in particular this holds for every $A \in \mathcal{B}$. If $x(A)$ is an element of $A \backslash U$ for each $A \in \mathcal{B}$, then $\{x(A)\}_{A \in \mathcal{B}}$ is a net associated to $\mathcal{F}$ in the same way as before, but $\{x(A)\}_{A \in \mathcal{B}}$ does not converge to x in X.

27 Cauchy filters

Let $(M, d(x, y))$ be a metric space. Remember that the diameter of a nonempty bounded subset E of M is defined by

$$\operatorname{diam} E = \sup\{d(x, y) : x, y \in E\}. \tag{27.1}$$

Note that the closure $\overline{E}$ is also a bounded set under these conditions, with the same diameter as E.

A filter $\mathcal{F}$ on M is said to be a Cauchy filter if for each $\epsilon > 0$ there is an $E \in \mathcal{F}$ such that

$$\operatorname{diam} E < \epsilon. \tag{27.2}$$

It is easy to see that convergent filters on M are Cauchy, and that Cauchy filters on complete metric spaces converge. More precisely, if $\mathcal{F}$ is a Cauchy filter on M, then there is a sequence $E_1, E_2, \ldots$ of elements of $\mathcal{F}$ such that $E_{j+1} \subseteq E_j$ and

$$\operatorname{diam} E_j < 1/j \tag{27.3}$$

for each j. If x_j is an element of E_j for each j, then it follows that $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence in M. If M is complete, then $\{x_j\}_{j=1}^\infty$ converges to an element x of M, and one can check that $\mathcal{F}$ also converges to x under these conditions.

Now let V be a topological vector space over the real or complex numbers. A filter $\mathcal{F}$ on V is said to be a Cauchy filter on V if for each open set U in V with $0 \in U$ there is an element E of $\mathcal{F}$ such that

$$E - E = \{y - z : y, z \in E\} \subseteq U. \tag{27.4}$$

This is equivalent to the previous definition of a Cauchy filter on a metric space when V is equipped with a translation-invariant metric that determines the same topology.

Suppose that a filter $\mathcal{F}$ on a topological vector space V converges to an element v of V, and let us check that $\mathcal{F}$ is a Cauchy filter on X. Let U be any open set in V that contains 0, and let W be an open set in V that contains 0 and satisfies

$$W - W \subseteq U, \tag{27.5}$$

as in Section 24. Thus $v + W$ is an open set in V that contains v, so that

$$v + W \in \mathcal{F}, \tag{27.6}$$

by the definition of convergence of a filter. Of course,

$$(v + W) - (v + W) = W - W, \tag{27.7}$$

so that (27.4) holds with $E = v + W$, as desired.

Let $\{v_a\}_{a \in A}$ be a net of elements of V, and let $\mathcal{F}$ be the filter associated to this net as in the previous section. Thus $\{v_a\}_{a \in A}$ converges to an element v of V if and only if $\mathcal{F}$ converges to v, as before. Similarly, $\{v_a\}_{a \in A}$ is a Cauchy net in V if and only if $\mathcal{F}$ is a Cauchy filter on V.

In the other direction, let $\mathcal{F}$ be a filter on V, and let $\mathcal{B}$ be a base for $\mathcal{F}$. As in the previous section, $\mathcal{B}$ is a directed system with respect to the ordering defined by reverse inclusion. If $\mathcal{F}$ is a Cauchy filter on V and $x(A)$ is an element of A for each $A \in \mathcal{B}$, then it is easy to see that $\{x(A)\}_{A \in \mathcal{B}}$ is a Cauchy net of elements of V indexed by $\mathcal{B}$. If $\mathcal{F}$ is a Cauchy filter on V, $x(A) \in A$ for every $A \in \mathcal{B}$, and $\{x(A)\}_{A \in \mathcal{B}}$ converges to an element x of V, then one can check that $\mathcal{F}$ also converges to x. It follows from the remarks in this and the preceding paragraph that every Cauchy net of elements of V converges to an element of V if and only if every Cauchy filter on V converges to an element of V, so that the definition of completeness of a topological vector space in Section 25 could just as well have been given in terms of Cauchy filters.

A filter $\mathcal{F}'$ on a set X is said to be a *refinement* of a filter $\mathcal{F}$ on X if $\mathcal{F} \subseteq \mathcal{F}'$ as collections of subsets of X. This is analogous to the notion of a subsequence of a sequence. If X is a topological space and $\mathcal{F}$ is a filter on X that converges to a point $p \in X$, then it is easy to see that every refinement of $\mathcal{F}$ converges to p as well.

If $\mathcal{F}$ is a Cauchy filter on a topological vector space V, and $\mathcal{F}'$ is a filter on V that is a refinement of $\mathcal{F}$, then it is easy to see that $\mathcal{F}'$ is also a Cauchy filter on V. If $\mathcal{F}$ is a Cauchy filter on V and $\mathcal{F}'$ is a refinement of $\mathcal{F}$ that converges to a vector $v \in V$, then one can check that $\mathcal{F}$ converges to v on V too. Note that the corresponding statements for Cauchy filters on metric spaces also hold. These statements are analogous to the classical facts that subsequences of Cauchy sequences are Cauchy sequences, and that a Cauchy sequence with a convergent subsequence converges to the same limit.

If $\mathcal{F}$ is a filter on a topological space X, and $\mathcal{F}'$ is a refinement of $\mathcal{F}$ that converges to a point $p \in X$, then $p \in \overline{E}$ for each $E \in \mathcal{F}'$, as in the previous section, and hence $p \in \overline{E}$ for each $E \in \mathcal{F}$. Conversely, if $p \in \overline{E}$ for each $E \in \mathcal{F}$, then there is a refinement $\mathcal{F}'$ of $\mathcal{F}$ that converges to p. More precisely, let $\mathcal{F}_0$ be the collection of subsets of X of the form $E \cap U$, where $E \in \mathcal{F}$ and U is an open set in X that contains p as an element. Under these conditions, one can check that $\mathcal{F}_0$ is a pre-filter on X, and that the filter $\mathcal{F}'$ that it generates is a refinement of $\mathcal{F}$ that converges to p on X. This gives a criterion for the convergence of Cauchy filters, by the remarks in the preceding paragraph.

28 Compactness

Let X be a topological space, and let K be a subset of X. Let us say that a collection $\{E_i\}_{i \in I}$ of closed subsets of X has the finite intersection property relative to K if

$$\Big(\bigcap_{j=1}^{n} E_{i_j}\Big) \cap K \neq \emptyset \tag{28.1}$$

for every finite subcollection $i_1, \ldots, i_n$ of indices in I. If K is compact, then this implies that

$$\Big(\bigcap_{i \in I} E_i\Big) \cap K \neq \emptyset. \tag{28.2}$$

Otherwise, if (28.2) does not hold, then the collection of open sets $U_i = X \backslash E_i$ would form an open covering of K in X, and the existence of a finite subcovering of K would contradict (28.1). Similarly, if every collection $\{E_i\}_{i \in I}$ of closed subsets of X with the finite intersection property relative to K satisfies (28.2), then one can reverse the argument to show that K is compact.

Now let $\mathcal{F}$ be a filter on X such that $K \in \mathcal{F}$. If $E_1, \ldots, E_n$ are finitely many elements of $\mathcal{F}$, then

$$\Big(\bigcap_{j=1}^{n} E_j\Big) \cap K \tag{28.3}$$

is also an element of $\mathcal{F}$, and hence is nonempty. In particular,

$$\Big(\bigcap_{j=1}^{n} \overline{E}_j\Big) \cap K \neq \emptyset, \tag{28.4}$$

so that the collection of closures $\overline{E}$ of elements E of $\mathcal{F}$ has the finite intersection property relative to K. If K is compact, then it follows that

$$\Big(\bigcap_{E \in \mathcal{F}} \overline{E}\Big) \cap K \neq \emptyset. \tag{28.5}$$

This is equivalent to saying that $\mathcal{F}$ has a refinement that converges to an element of K, as discussed at the end of the previous section.

Conversely, let $\{E_i\}_{i \in I}$ be a collection of closed subsets of X with the finite intersection property relative to $K \subseteq X$. Let $\mathcal{F}_0$ be the collection of subsets of X of the form

$$\Big(\bigcap_{j=1}^{n} E_{i_j}\Big) \cap K, \tag{28.6}$$

where $i_1, \ldots, i_n$ are finitely many elements of I. Note that $\mathcal{F}_0$ is a pre-filter on X, and let $\mathcal{F}$ be the filter on X generated by $\mathcal{F}$. Observe also that

$$K \in \mathcal{F} \text{ and } E_i \in \mathcal{F} \text{ for each } i \in I, \tag{28.7}$$

by construction. If $\mathcal{F}$ has a refinement that converges to an element of K, then (28.5) holds, by the discussion at the end of the previous section. In particular, this implies that (28.2) holds, because E_i is a closed subset of X and an element of $\mathcal{F}$ for each $i \in I$. It follows that $K \subseteq X$ is compact when every filter $\mathcal{F}$ on X that contains K as an element has a refinement that converges to an element of K, by the characterization of compactness mentioned at the beginning of the section.

Let V be a topological vector space over the real or complex numbers, let $\mathcal{F}$ be a Cauchy filter on V, and let K be a compact subset of V which is also an element of $\mathcal{F}$. The preceding discussion implies that $\mathcal{F}$ has a refinement that converges to an element p of K, and hence that $\mathcal{F}$ converges to p. Of course, the corresponding statement for metric spaces instead of topological vector spaces also holds. This is analogous to the classical fact that a Cauchy sequence $\{x_j\}_{j=1}^{\infty}$ of elements of a compact subset K of a metric space M has a subsequence that converges to an element p of K, and hence $\{x_j\}_{j=1}^{\infty}$ converges to p too.

29 Ultrafilters

A filter $\mathcal{F}$ on a set X is said to be an *ultrafilter* if it is maximal with respect to refinement, in the sense that if $\mathcal{F}'$ is a filter on X that is a refinement of $\mathcal{F}$, then $\mathcal{F}' = \mathcal{F}$. If $p \in X$, then one can check that

$$\mathcal{F}_p = \{A \subseteq X : p \in A\} \tag{29.1}$$

is an ultrafilter on X, for instance. If $\mathcal{F}$ is any filter on X, then one can use the axiom of choice through Zorn's lemma or the Hausdorff maximality principle to show that there is an ultrafilter $\widetilde{\mathcal{F}}$ on X which is a refinement of $\mathcal{F}$.

If X is a topological space, $K \subseteq X$ is compact, and $\mathcal{F}$ is an ultrafilter on X that contains K as an element, then $\mathcal{F}$ converges on X to an element of K. This is because any filter $\mathcal{F}$ on X that contains K as an element has a refinement $\mathcal{F}'$ that converges to an element of K, as in the previous section, and because $\mathcal{F}' = \mathcal{F}$ when $\mathcal{F}$ is an ultrafilter. Conversely, suppose that a set $K \subseteq X$ has the property that every ultrafilter on X that contains K as an element converges to an element of K, and let us check that K is compact. To see this, it suffices to show that any filter $\mathcal{F}$ on X that contains K as an element has a refinement that converges to an element of K, by the discussion in the previous section. If $\mathcal{F}'$ is a refinement of $\mathcal{F}$ that is an ultrafilter on X, then K is also an element of $\mathcal{F}'$, so that $\mathcal{F}'$ converges to an element of K by hypothesis, as desired.

Let $\mathcal{F}$ be a filter on a set X, and let us say that a subset E of X has property $P_{\mathcal{F}}$ if

$$A \cap E \neq \emptyset \tag{29.2}$$

for every $A \in \mathcal{F}$. In this case, it is easy to see that

$$\mathcal{F}_0 = \{A \cap E : A \in \mathcal{F}\} \tag{29.3}$$

is a pre-filter on X, and that the filter $\mathcal{F}'$ generated by $\mathcal{F}_0$ is a refinement of $\mathcal{F}$. If $\mathcal{F}$ is an ulrafilter, then it follows that $\mathcal{F}' = \mathcal{F}$, and hence that $E \in \mathcal{F}$, because $E \in \mathcal{F}'$ by construction. Conversely, if $\mathcal{F}$ is a filter on X, $\mathcal{F}'$ is another filter on X which is a refinement of $\mathcal{F}$, and $E \in \mathcal{F}'$, then E has property $P_{\mathcal{F}}$. This is because $A \cap E \in \mathcal{F}'$ when $A \in \mathcal{F} \subseteq \mathcal{F}'$ and $E \in \mathcal{F}'$, so that (29.2) holds. Thus $\mathcal{F}' = \mathcal{F}$ when $\mathcal{F}'$ is a refinement of $\mathcal{F}$ and $\mathcal{F}$ contains every set $E \subseteq X$ that satisfies property $P_{\mathcal{F}}$. This shows that a filter $\mathcal{F}$ on X is an ultrafilter if and only if $\mathcal{F}$ contains every set $E \subseteq X$ that satisfies property $P_{\mathcal{F}}$.

Let $\mathcal{F}$ be an ultrafilter on a set X, and let E be a subset of X. If E has the property $P_{\mathcal{F}}$ discussed in the previous paragraph, then $E \in \mathcal{F}$, as before. Otherwise, there is an $A \in \mathcal{F}$ such that

$$A \cap E = \emptyset, \tag{29.4}$$

in which case $A \subseteq X \backslash E$ and hence $X \backslash E \in \mathcal{F}$. Thus for each set $E \subseteq X$,

$$\text{either } E \in \mathcal{F} \text{ or } X \backslash E \in \mathcal{F} \tag{29.5}$$

when $\mathcal{F}$ is an ultrafilter on X. Conversely, if $\mathcal{F}$ is a filter on X such that (29.5) holds for every $E \subseteq X$, then $\mathcal{F}$ is an ultrafilter on X, because (29.5) implies that $E \in \mathcal{F}$ when $E \subseteq X$ satisfies the property $P_{\mathcal{F}}$ discussed in the previous paragraph.

Let $\mathcal{F}$ be an ultrafilter on a set X again, and let A be a subset of X which is an element of $\mathcal{F}$. Also let $B_1, \ldots, B_n$ be finitely many subsets of X such that

$$A \subseteq \bigcup_{j=1}^{n} B_j. \tag{29.6}$$

Under these conditions,

$$B_j \in \mathcal{F} \tag{29.7}$$

for at least one j, $1 \le j \le n$. Otherwise, if $B_j \notin \mathcal{F}$ for each $j = 1, \ldots, n$, then we would have that $X \backslash B_j \in \mathcal{F}$ for each j, as in the previous paragraph. This would imply that

$$X \backslash \Big(\bigcup_{j=1}^{n} B_j \Big) = \bigcap_{j=1}^{n} (X \backslash B_j) \in \mathcal{F}, \tag{29.8}$$

contradicting (29.6) and the hypothesis that $A \in \mathcal{F}$.

Now let V be a topological vector space over the real or complex numbers, let K be a totally bounded subset of V, and let $\mathcal{F}$ be an ultrafilter on V that contains K as an element. We would like to show that $\mathcal{F}$ is a Cauchy filter on V under these conditions. Let U be an open set in V that contains 0 as an element, and let W be another open set in V such that $0 \in W$ and $W - W \subseteq U$. Because K is totally bounded, there are finitely many vectors $v_1, \ldots, v_n$ in V such that

$$K \subseteq \bigcup_{j=1}^{n} (v_j + W). \tag{29.9}$$

It follows from the remarks in the previous paragraph that

$$v_j + W \in \mathcal{F} \tag{29.10}$$

for some j, because $\mathcal{F}$ is an ultrafilter and $K \in \mathcal{F}$. Of course,

$$(v_j + W) - (v_j + W) = W - W \subseteq U, \tag{29.11}$$

by construction. Thus $\mathcal{F}$ is a Cauchy filter on V, as desired.

Suppose that $K \subseteq V$ has the completeness property that every Cauchy filter on V that contains K as an element converges to an element of K. This is equivalent to asking that every Cauchy net of elements of K converge to an element of K. If K is also totally bounded, then the argument in the previous paragraph would imply that every ultrafilter on V that contains K as an element is a Cauchy filter, and hence converges to an element of K. This shows that totally bounded subsets of V with the completeness property just mentioned are compact, using the characterization of compactness in terms of ultrafilters discussed earlier in the section. Conversely, compact subsets of V are totally bounded, and it is easy to see that they have this completeness property too. More precisely, if $K \subseteq V$ is compact and $\mathcal{F}$ is a filter on V that contains K as an element, then $\mathcal{F}$ has a refinement that converges to an element of K, as in Section 28. If $\mathcal{F}$ is actually a Cauchy filter on V, then it follows that $\mathcal{F}$ converges to the same element of K, as desired.

30 A technical point

Let V be a topological vector space over the real or complex numbers, and let $\mathcal{B}$ be a local base for the topology of V at 0. As usual, $\mathcal{B}$ may be considered as a directed system with respect to the ordering which is the reverse of inclusion. Let $\mathcal{F}$ be a Cauchy filter on V, so that for each $U \in \mathcal{B}$ there is a set $E(U) \in \mathcal{F}$ such that

$$E(U) - E(U) \subseteq U. \tag{30.1}$$

If $x(U) \in E(U)$ for each $U \in \mathcal{B}$, then $\{x(U)\}_{U \in \mathcal{B}}$ is a Cauchy net in V indexed by $\mathcal{B}$. To see this, let U_0 be an open set in V that contains 0, and let U_1 be an element of $\mathcal{B}$ such that

$$U_1 + U_1 \subseteq U_0. \tag{30.2}$$

The existence of U_1 follows from the continuity of addition on V at 0, and the fact that $\mathcal{B}$ is a local base for the topology of V at 0. We would like to check that

$$x(U') - x(U'') \in U_0 \tag{30.3}$$

for every $U', U'' \in \mathcal{B}$ such that $U', U'' \subseteq U_1$.

If U', U'' are any two elements of $\mathcal{B}$, then

$$E(U') \cap E(U'') \neq \emptyset, \tag{30.4}$$

because $E(U') \cap E(U'') \in \mathcal{F}$. If $y(U', U'')$ is an element of $E(U') \cap E(U'')$, then we get that

$$x(U') - x(U'') = (x(U') - y(U', U'')) + (y(U', U'') - x(U'')) \tag{30.5}$$

is an element of

$$(E(U') - E(U')) + (E(U'') - E(U'')) \subseteq U' + U''. \tag{30.6}$$

Hence

$$x(U') - x(U'') \in U' + U'' \subseteq U_1 + U_1 \subseteq U_0 \tag{30.7}$$

when $U', U'' \subseteq U_1$, as desired. This shows that $\{x(U)\}_{U \in \mathcal{B}}$ is a Cauchy net in V, where $\mathcal{B}$ is ordered by reverse inclusion.

If $\{x(U)\}_{U \in \mathcal{B}}$ converges to an element x of V, then one can check that $\mathcal{F}$ also converges to x. This is a bit different from the analogous statement in Section 27, where the convergence of a Cauchy filter was reduced to the convergence of a Cauchy net index by a base for the filter. If there is a countable local base for the topology of V at 0, for instance, then one can take $\mathcal{B}$ to be countable and linearly ordered by inclusion, to reduce the convergence of a Cauchy filter on V to the convergence of a Cauchy sequence in V.

Similarly, if $\{v_a\}_{a \in A}$ is a Cauchy net of elements of V indexed by any directed system A, then one can get a Cauchy net indexed by $\mathcal{B}$ whose convergence in V implies the convergence of $\{v_a\}_{a \in A}$. One can apply the previous argument to the filter associated to $\{v_a\}_{a \in A}$ as in Section 26, or argue a bit more directly and take for each $U \in \mathcal{B}$ an index $a(U)$ in A such that $v_b - v_c \in U$ for every $b, c \in A$ with $a(U) \prec b, c$. This leads to the net $\{v_{a(U)}\}_{U \in \mathcal{B}}$ indexed by $\mathcal{B}$, which is a Cauchy net for essentially the same reasons as before, and the convergence of which implies the convergence of $\{v_a\}_{a \in A}$, to the same limit. If there is a countable local base for the topology of V at 0, then one can take $\mathcal{B}$ to be countable and linearly ordered, to get a Cauchy sequence of elements of V whose convergence implies the convergence of $\{v_a\}_{a \in A}$. This fact was mentioned previously in Section 24, and the present discussion gives a more general version of this, without using metrics.

31 Dual spaces

Let W be a finite-dimensional vector space over the real or complex numbers. Thus the algebraic dual space W^* of linear functionals on W is also a finite-dimensional vector space, with the same dimension as W. If Z is a linear subspace of W^* which is not equal to W^*, then there is a nonzero vector $w \in W$ such that $\lambda(w) = 0$ for every $\lambda \in Z$. More precisely, the collection of $w \in W$ such that $\lambda(w) = 0$ for every $\lambda \in Z$ is a linear subspace of W, whose dimension is equal to the dimension of W minus the dimension of Z.

Now let V be a topological vector space over the real or complex numbers, and suppose that for each nonzero vector $v \in V$ there is a continuous linear functional λ on V such that $\lambda(v) \neq 0$. Let W be a finite-dimensional linear subspace of V, and remember that V' denotes the vector space of continuous linear functionals on V. If Z is the linear subspace of W^* consisting of restrictions of continuous linear functionals on V to W, then it follows from the remarks in the previous paragraph that $Z = W^*$. This implies that V' is a dense linear subspace of the algebraic dual V^* of V with respect to the weak* topology under these conditions.

Of course, V' is normally a proper linear subspace of V^* when V is infinite-dimensional, although we have seen some situations where $V' = V^*$. If V is any real or complex vector space, then one can consider the weak topology on V associated to the space V^* of all linear functionals on V, as in Section 15. In this case, every linear functional on V is continuous by construction. Alternatively, one can consider the topology on V determined by the collection of all seminorms on V, which also came up in Section 21.

In particular, the topological dual V' of an infinite-dimensional topological vector space V is normally not complete in the sense of Section 25 with respect to the weak* topology. However, V' is sequentially complete with respect to the weak* topology when V is an F-space, which is to say that V is metrizable and sequentially complete. If V is any real or complex vector space, then a sequence $\{\lambda_j\}_{j=1}^\infty$ of linear functionals on V is a Cauchy sequence with respect to the weak* topology if and only if $\{\lambda_j\}_{j=1}^\infty$ is a Cauchy sequence in $\mathbf{R}$ or $\mathbf{C}$, as appropriate, for each $v \in V$. This implies that $\{\lambda_j(v)\}_{j=1}^\infty$ converges to a real or complex number $\lambda(v)$ for each $v \in V$, because of the completeness of the real and complex numbers. As usual, it is easy to see that $\lambda(v)$ defines a linear functional on V under these conditions. Note that $\{\lambda_j(v)\}_{j=1}^\infty$ is in particular a bounded sequence of real or complex numbers for each $v \in V$. If V is an F-space and λ_j is a continuous linear functional on V for each j, then it follows that there is an open set U in V with $0 \in U$ such that the λ_j's are uniformly bounded on U, as in Section 18. This implies that λ is uniformly bounded on U, and hence that λ is continuous on V, as desired.

32 Bounded linear mappings

Let V and W be topological vector spaces, both real or both complex. A linear mapping T from V into W is said to be *bounded* if for each bounded set $E \subseteq V$, $T(E)$ is a bounded set in W. It is easy to see that continuous linear mappings are bounded. Let $\mathcal{CL}(V, W)$ denote the space of continuous linear mappings from V into W, and let $\mathcal{BL}(V, W)$ be the space of bounded linear mappings from V into W, so that

$$\mathcal{CL}(V, W) \subseteq \mathcal{BL}(V, W). \tag{32.1}$$

Of course, $\mathcal{CL}(V, W)$ is a vector space with respect to pointwise addition and scalar multiplication, since the sum of two continuous linear mappings is also continuous, as are scalar multiples of continuous mappings. One can check that the analogous statements hold for bounded linear mappings as well, using the fact that the sum of two bounded sets is bounded, as in Section 4. It follows that $\mathcal{BL}(V, W)$ is a vector space with respect to pointwise addition and scalar multiplication, and that $\mathcal{CL}(V, W)$ is a linear subspace of $\mathcal{BL}(V, W)$. Note that compositions of bounded linear mappings are bounded, as in the case of continuous mappings.

Let us say that a linear mapping $T : V \to W$ is strongly bounded if there is an open set U in V with $0 \in U$ such that $T(U)$ is a bounded set in W. It is easy to see that strongly bounded mappings are continuous. If the topology on V is determined by a norm, then every bounded linear mapping from V into W is obviously strongly bounded, and hence continuous. This works more generally when there is a bounded open set in V that contains 0, such as an L^p space for any $p > 0$. Similarly, if the topology on W is determined by a norm, or more generally if there is a bounded open set in W that contains 0, then every continuous linear mapping from V into W is strongly bounded.

There is a nice theorem that states that a bounded linear mapping T from V into W is continuous when there is a countable local base for the topology of V at 0. In this case, it suffices to show that T is sequentially continuous at 0. Let $\{v_j\}_{j=1}^\infty$ be a sequence of vectors in V that converges to 0. A key lemma implies that there is a sequence $\{r_j\}_{j=1}^\infty$ of positive real numbers such that $\{r_j\}_{j=1}^\infty$ converges to 0 in $\mathbf{R}$ and $\{r_j^{-1}\, v_j\}_{j=1}^\infty$ converges to 0 in V. In particular, the set K consisting of $r_j^{-1}\, v_j$ for $j \in \mathbf{Z}_+$ and 0 is compact, and hence bounded. If T is bounded, then it follows that $T(K)$ is bounded in W. Under these conditions, it is easy to see that

$$T(v_j) = r_j\, T(r_j^{-1}\, v_j) \to 0 \tag{32.2}$$

in W as $j \to \infty$, as desired, because $\{r_j\}_{j=1}^\infty$ converges to 0 in $\mathbf{R}$ and $T(K)$ is bounded in W.

As in Section 12, the hypothesis that V have a countable local base for its topology at 0 implies that there is a translation-invariant metric $d(v, w)$ on V that determines the same topology on V. In this case, one can check that

$$d(n\,v, 0) \le n\,d(v, 0) \tag{32.3}$$

for each positive integer n. More precisely, this uses the fact that

$$d((j+1)\,v, j\,v) = d(v, 0) \tag{32.4}$$

for each positive integer j, by translation-invariance of the metric. The key lemma mentioned in the previous paragraph can easily be verified using this inequality, although it can also be derived more directly from the existence of a countable local base for the topology of V at 0.

33 Bounded linear functionals

Let V be a topological vector space over the real or complex numbers, and let $V^\flat$ be the vector space of bounded linear functionals on V. More precisely, $V^\flat$ is the same as the space of bounded linear mappings from V into $\mathbf{R}$ or $\mathbf{C}$, as appropriate. Thus $V^\flat$ may be considered as a linear subspace of the algebraic dual V^* of all linear functionals on V, and $V^\flat$ contains the topological dual V' of continuous linear functionals on V as a linear subspace. If V has a countable local base for its topology at 0, then $V' = V^\flat$, as in the previous section.

If $E \subseteq V$ is a nonempty bounded set, and λ is a bounded linear functional on V, then λ is bounded on E, and we put

$$N_E^\flat(\lambda) = \sup_{v \in E} |\lambda(v)|. \tag{33.1}$$

If E consists of a single element v, then this reduces to the corresponding seminorm $N_v^*(\lambda)$ used in Section 16 to define the weak* topology. In particular, if $\lambda \ne 0$, then there is a $v \in V$ such that $\lambda(v) \ne 0$, and hence $N_E^\flat(\lambda) \ne 0$ when $v \in E$. Thus the collection of all of these seminorms $N_E^\flat(\lambda)$ associated to nonempty bounded sets $E \subseteq V$ is a nice collection of seminorms on $V^\flat$, which defines a topology on $V^\flat$ that makes $V^\flat$ into a topological vector space, as in Section 9. This topology on $V^\flat$ is at least as strong as the weak* topology, since this collection of seminorms contains the seminorms used to define the weak* topology.

If the topology on V is determined by a norm N, then one can take E to be the unit ball in V associated to N, and $N_E^\flat(\lambda)$ is the usual dual norm on $V^\flat = V'$ corresponding to N. More generally, if U is a bounded open set in V that contains 0, then $N_U^\flat(\lambda)$ is a norm on $V^\flat = V'$. In this case, if $E \subseteq V$ is any nonempty bounded set, then $E \subseteq t\,U$ when $|t|$ is sufficiently large, which implies that

$$N_E^\flat(\lambda) \le C\,N_U^\flat(\lambda) \tag{33.2}$$

for some $C \geq 0$ and every $\lambda \in V^\flat$. It follows that the topology on $V^\flat$ determined by $N_U^\flat$ is the same as the topology determined by all of these seminorms $N_E^\flat$ when $U \subseteq V$ is a bounded open set with $0 \in U$.

If V is any topological vector space, then $V^\flat$ is complete with respect to the topology determined by the collection of seminorms $N_E^\flat$, in the sense of Section 25. To see this, one can begin with a Cauchy net or filter in $V^\flat$ with respect to this topology, which one would like to show converges to an element of $V^\flat$ with respect to this topology. This net or filter automatically satisfies the Cauchy condition with respect to the weak* topology too, since the topology on $V^\flat$ determined by the seminorms $N_E^\flat$ is at least as strong as the weak* topology. Hence the Cauchy net or filter converges pointwise on V to a linear functional λ on V, as discussed previously. Because the Cauchy condition is actually satisfied with respect to the topology on $V^\flat$ determined by the seminorms $N_E^\flat$, one can check that the Cauchy net or filter converges to λ uniformly on bounded subsets of V. In particular, this implies that λ is a bounded linear functional on V, since the linear functionals on V in the Cauchy net or filter are bounded by hypothesis. The remaining point is that the Cauchy net or filter converges to λ with respect to the topology on $V^\flat$ determined by the seminorms $N_E^\flat$, which is not difficult to verify.

34 The dual norm

Let V be a real or complex vector space, and let N be a norm on V. Thus V is a topological vector space with respect to the topology determined by N, and bounded linear functionals on V are the same as continuous linear functionals, so that $V^\flat = V'$. The dual norm N' on V' corresponding to N is defined by

$$N'(\lambda) = \sup\{|\lambda(v)| : v \in V,\ N(v) \leq 1\}. \tag{34.1}$$

It is easy to see that this defines a norm on V', which is the same as $N_E^\flat$ in (33.1), with E equal to the closed unit ball in V with respect to N. One can also check that V' is complete with respect to the metric corresponding to N'. Although this may be treated as a special case of the discussion of completeness at the end of the previous section, it is somewhat simpler, because one only needs to consider Cauchy sequences in V'. As usual, if $\{\lambda_j\}_{j=1}^\infty$ is a Cauchy sequence in V' with respect to N', then one can first observe that $\{\lambda_j(v)\}_{j=1}^\infty$ is a Cauchy sequence in $\mathbf{R}$ or $\mathbf{C}$, as appropriate, for each $v \in V$. Hence $\{\lambda_j(v)\}_{j=1}^\infty$ converges to a real or complex number $\lambda(v)$ for each $v \in V$, and one can then verify that λ is a bounded linear functional on V, and that $\{\lambda_j\}_{j=1}^\infty$ converges to λ with respect to N'.

As in the previous section, the topology on V' determined by N' is at least as strong as the weak* topology on V'. In particular, bounded subsets of V' with respect to N' are also bounded with respect to the weak* topology. If V is complete, then bounded subsets of V' with respect to the weak* topology are also bounded with respect to N', as in Section 18. If V is infinite-dimensional, then the Hahn–Banach theorem implies that V' is also infinite-dimensional, and it is easy to see that the topology on V' determined by the dual norm N' is strictly stronger than the weak* topology on V'. It follows that the identity mapping on V' is bounded as a mapping from V' with the weak* topology into V' with the topology determined by N', and not continuous, when V is infinite-dimensional and complete.

Similarly, the bounded linear functionals on V' with respect to the weak* topology are the same as the bounded linear functionals on V' with respect to N' when V is complete, since the bounded subsets of V' with respect to the two topologies are the same. If $v \in V$, then

$$L_v(\lambda) = \lambda(v) \tag{34.2}$$

defines a linear functional on V' which is continuous with respect to the weak* topology, and hence also with respect to the topology on V' determined by N'. Every continuous linear functional on V' with respect to the weak* topology is of this form, as in Section 15. Every bounded linear functional on V' with respect to the topology determined by N' is continuous with respect to this topology as well. However, there may be bounded linear functionals on V' with respect to N' which are not of the form L_v for some $v \in V$.

One of the simplest situations where this occurs is with $V = c_0$, the space of sequences of real or complex numbers that converges to 0, with respect to the supremum norm. In this case, the dual space V' can be identified with ℓ^1, the space of sequences of real or complex numbers that are absolutely summable. The dual of ℓ^1 can then be identified with the space ℓ^∞ of bounded sequences of real or complex numbers, with the supremum norm. The mapping from $v \in V$ to a linear functional L_v on V' corresponds to the standard inclusion of c_0 in ℓ^∞ with respect to these identifications, and of course c_0 is a proper linear subspace of ℓ^∞.

35 The second dual

Let V be a real or complex vector space again, and let N be a norm on V. As in the previous sections, the space $V^\flat$ of bounded linear functionals on V is the same as the space V' of continuous linear functionals on V, and we let N' be the dual norm on $V^\flat = V'$ associated to N on V. Thus V' is now a vector space with a norm N', and we can repeat the process to get a vector space V'' with a norm N''. More precisely, V'' is the space of bounded linear functionals on V' with respect to the dual norm N', which is the same as the space of continuous linear functionals on V' with respect to the topology on V' determined by V'. If L is a bounded linear functional on V' with respect to N', then the dual norm of L with respect to N' is given by

$$N''(L) = \sup\{|L(\lambda)| : \lambda \in V',\, N'(\lambda) \le 1\}. \tag{35.1}$$

If $v \in V$, then $L_v(\lambda) = \lambda(v)$ defines a bounded linear functional on V', as in the preceding section. Observe that

$$|L_v(\lambda)| = |\lambda(v)| \le N'(\lambda)\, N(v) \tag{35.2}$$

for each $\lambda \in V'$, by the definition (34.1) of the dual norm N'. This implies that

$$N''(L_v) \le N(v) \tag{35.3}$$

for each $v \in V$. Using the Hahn–Banach theorem, one can show that

$$N''(L_v) = N(v) \tag{35.4}$$

for every $v \in V$, by finding $\lambda \in V'$ for which equality holds in (35.2).

Remember that the weak topology on V is defined as in Section 15, using the space V' of continuous linear functionals on V with respect to V. Basically, the weak topology on V is the weakest topology on V such that the elements of V' are continuous. In particular, every open set in V with respect to the weak topology is also an open set with respect to the topology determined by N, and we mentioned earlier that the weak topology is strictly weaker than the topology determined by N when V is infinite-dimensional. It follows that bounded subsets of V with respect to N are automatically bounded with respect to the weak topology, and one can check that the converse holds too. Basically, one can use the embedding of V into V'' described in the previous paragraph to reduce this to the analogous statement for the weak* topology discussed in the preceding section, applied to V' instead of V. Thus it is the completeness of V' with respect to the dual norm N' that is important in this case, rather than the completeness of V. As a consequence, we get that the identity mapping on V is bounded as a mapping from V with the weak topology into V with the topology determined by the norm, even though this mapping is not continuous when V is infinite-dimensional.

36 Bounded sequences

Let V be a topological vector space over the real or complex numbers. If $E \subseteq V$ is not bounded, then there is an open set U in V with $0 \in U$ such that for each positive integer n there is an element v_n of E that is not contained in $n\,U$. In particular, the set of v_n's is itself not bounded in this case. It follows that $E \subseteq V$ is bounded if and only if every countable subset of E is bounded.

If $E \subseteq V$ is bounded, $\{v_j\}_{j=1}^\infty$ is a sequence of elements of E, and $\{t_j\}_{j=1}^\infty$ is a sequence of real or complex numbers that converges to 0, then $\{t_j\,v_j\}_{j=1}^\infty$ converges to 0 in V. This is easy to see, and it was also mentioned in Section 32. Conversely, suppose that $E \subseteq V$ has the property that for each sequence $\{v_j\}_{j=1}^\infty$ of elements of E and every sequence $\{t_j\}_{j=1}^\infty$ of real or complex numbers that converges to 0, $\{t_j\,v_j\}_{j=1}^\infty$ converges to 0 in V. One can check that E is bounded in V under these conditions, using the remarks in the previous paragraph.

Now let W be another topological vector space, which is real if V is real, and complex if V is complex. Let T be a linear mapping from V into W, and suppose that T is sequentially continuous at 0. More precisely, if $\{u_j\}_{j=1}^\infty$ is a sequence of elements of V that converges to 0, then the sequential continuity of T at 0 means that $\{T(u_j)\}_{j=1}^\infty$ converges to 0 in W. This implies that T

is sequentially continuous at every point in V, because of linearity. Sequential continuity is implied by ordinary continuity, and the converse holds when there is a countable local base for the topology of V at 0.

It is easy to see that sequentially continuous linear mappings are bounded, using the characterization of bounded sets in terms of convergent sequences mentioned earlier. This extends the fact that continuous linear mappings are bounded, as in Section 32. However, there are examples of bounded linear mappings that are not sequentially continuous, as in the previous two sections.

37 Continuous extensions

Let V and W be topological vector spaces, both real or both complex, let Z be a dense linear subspace of V, and let T be a continuous linear mapping from Z into W. If W is complete, then there is a unique extension of T to a continuous linear mapping from V into W. Of course, uniqueness follows from continuity and the fact that the range W is Hausdorff.

To get the existence of the extension, let v be any element of V, and let $\{v_\alpha\}_{\alpha \in A}$ be a net of elements of Z that converges to v in V. In particular, this is a Cauchy net in Z, with respect to the topology on Z induced by the one on V. One can check that $\{T(v_\alpha)\}_{\alpha \in A}$ is a Cauchy net in W, using the continuity and linearity of T. If W is complete, then it follows that $\{T(v_\alpha)\}_{\alpha \in A}$ converges to an element of W. One can also check that the limit of $\{T(v_\alpha)\}_{\alpha \in A}$ in W does not depend on the choice of net $\{v_\alpha\}_{\alpha \in A}$ of elements of Z converging to v, and we define $T(v)$ to be the common value of the limit. It remains to verify that this extension of T to V is linear and continuous, which is not difficult to do. Note that one can take A to be a local base for the topology of V at 0 here, ordered by reverse inclusion.

Alternatively, for each $v \in V$, let $\mathcal{F}_{v,Z}$ be the collection of $E \subseteq Z$ such that

$$(v + U) \cap Z \subseteq E \tag{37.1}$$

for some open set U in V that contains 0. It is easy to see that this is a filter on Z, and in fact a Cauchy filter on Z. More precisely, $\mathcal{F}_{v,Z}$ can also be considered as a pre-filter on V, which generates a filter $\mathcal{F}'_{v,Z}$ on V that converges to v. This implies that $\mathcal{F}'_{v,Z}$ is a Cauchy filter on V, and hence that $\mathcal{F}_{v,Z}$ is a Cauchy filter on Z, with respect to the topology on Z induced by the one on V. Let $T_*(\mathcal{F}_{v,Z})$ be the collection of subsets B of W such that

$$T^{-1}(B) \in \mathcal{F}_{v,Z}. \tag{37.2}$$

One can check that this is a filter on W, and in fact a Cauchy filter on W, using the continuity and linearity of T. If W is complete, then $T_*(\mathcal{F}_{v,Z})$ converges to an element $T(v)$ of W. This defines an extension of T of V, which is equivalent to the one described earlier using nets. It is not difficult to verify that this extension is linear and continuous, as before.

38 Sublinear functions

Let V be a vector space over the real or complex numbers. A real-valued function $p(v)$ on V is said to be *sublinear* if

$$p(t\,v) = t\,p(v) \tag{38.1}$$

for every $v \in V$ and nonnegative real number t, and

$$p(v + w) \le p(v) + p(w) \tag{38.2}$$

for every $v, w \in V$. Thus seminorms on V are sublinear, as are linear functionals on V when V is real, or real parts of linear functionals on V when V is complex.

If p is a sublinear function on V, then $p(0) = 0$, and hence

$$p(v) + p(-v) \ge 0 \tag{38.3}$$

for every $v \in V$. If V is a real vector space and

$$p(-v) = p(v) \tag{38.4}$$

for every $v \in V$, then it follows that $p(v) \ge 0$ for every $v \in V$, and that $p(v)$ is a seminorm on V. Similarly, if V is a complex vector space and

$$p(a\,v) = p(v) \tag{38.5}$$

for every $a \in \mathbf{C}$ with $|a| = 1$, then $p(v)$ is a seminorm on V.

Suppose that $U \subseteq V$, $0 \in U$, and U is star-like about 0, in the sense that

$$t\,u \in U \tag{38.6}$$

for every $u \in U$ and real number t such that $0 \le t \le 1$. Equivalently, $t\,U \subseteq U$ when $0 \le t \le 1$. Suppose also that U has the absorbing property that for each $v \in V$ there is an $t(v) > 0$ such that

$$t(v)\,v \in U, \tag{38.7}$$

which implies that $t\,v \in U$ when $0 \le t \le t(v)$, because U is star-like about 0. The *Minkowski functional* on V associated to U is defined by

$$p(v) = \inf\{r > 0 : r^{-1}\,v \in U\} = \inf\{r > 0 : v \in r\,U\}, \tag{38.8}$$

which makes sense because of the absorbing property of U. Note that $p(v)$ automatically satisfies the homogeneity condition (38.1). If $-U = U$, then $p(v)$ satisfies (38.4) for every $v \in V$ as well. If V is a complex vector space and $a\,U = U$ for every $a \in \mathbf{C}$ with $|a| = 1$, then $p(v)$ satisfies (38.5).

If $u \in U$, then clearly $p(u) \le 1$. If V is a topological vector space and U is also an open set in V, then $p(u) < 1$ for every $u \in U$, because for each $u \in U$ we have that $t\,u \in U$ when t is sufficiently close to 1. Conversely, if $v \in V$ satisfies

$p(v) < 1$, then $R\,v \in U$ for some $R > 1$, and hence $v \in U$, because U is star-like about 0. If U is convex, then p satisfies the subadditivity condition (38.2), and is therefore sublinear. More precisely, if $v, w \in V$, $p(v) < r$, $p(w) < t$, then $r^{-1}\,v$ and $t^{-1}\,w$ are contained in U, by the definition of p, and because U is star-like about 0. If U is convex, then it follows that

$$(r+t)^{-1}\,(v+w) = \frac{r}{r+t}\,(r^{-1}\,v) + \frac{t}{r+t}\,(t^{-1}\,w) \in U. \tag{38.9}$$

This implies that $p(v+w) \le r+t$, and hence that (38.2) holds, because $r > p(v)$ and $t > p(w)$ are arbitrary.

39 Hahn–Banach, revisited

Let V be a vector space over the real numbers, and let $p(v)$ be a sublinear function on V. Suppose that W is a linear subspace of V, and that λ is a linear functional on W that satisfies

$$\lambda(v) \le p(v) \tag{39.1}$$

for every $v \in W$. Under these conditions, the theorem of Hahn and Banach states that λ can be extended to a linear functional on V that satisfies (39.1) for every $v \in V$. If $p(v)$ satisfies (38.4), then (39.1) implies that

$$-\lambda(v) = \lambda(-v) \le p(-v) = p(v), \tag{39.2}$$

and hence that

$$|\lambda(v)| \le p(v). \tag{39.3}$$

In this case, this version of the Hahn–Banach theorem reduces to the one for seminorms in Section 14.

Now let V be a topological vector space over the real numbers, and let U be a convex open set in V that contains 0. Thus the corresponding Minkowski functional (38.8) is sublinear, as before. Let v_0 be any element of V, and let W_0 be the 1-dimensional linear subspace of V spanned by v_0. If λ_0 is the linear functional on W_0 defined by

$$\lambda_0(t\,v_0) = t\,p(v_0) \tag{39.4}$$

for each $t \in \mathbf{R}$, then

$$\lambda_0(t\,v) = t\,p(v_0) = p(t\,v_0) \tag{39.5}$$

when $t \ge 0$, and

$$\lambda_0(t\,v_0) = t\,p(v_0) \le 0 \le p(t\,v_0) \tag{39.6}$$

when $t \le 0$.

The Hahn–Banach theorem implies that there is an extension of λ_0 to a linear functional on V that satisfies

$$\lambda_0(v) \le p(v) \tag{39.7}$$

for every $v \in V$. In particular, $\lambda_0(u) < 1$ when $u \in U$, because $p(u) < 1$ when $u \in U$. Similarly, $-\lambda_0(v) = \lambda_0(-v) < 1$ when $v \in -U$, so that $\lambda_0(v) > -1$ when $v \in -U$. It follows that

$$|\lambda_0(v)| < 1 \tag{39.8}$$

for every $v \in U \cap (-U)$, which implies that λ_0 is continuous on V, since $U \cap (-U)$ is an open set in V that contains 0. Note that $p(v_0) \geq 1$ when $v_0 \in V \backslash U$, so that $\lambda_0(u) < \lambda_0(v_0)$ for every $u \in U$ in this case.

Suppose now that E is a nonempty closed convex set in a locally convex topological vector space V over the real numebers, and that $v_1 \in V \backslash E$. Because E is closed and V is locally convex, there is a convex open set U_1 in V such that $v_1 \in U_1$ and $U_1 \subseteq V \backslash E$. Put $U_2 = E - U_1$, so that U_2 is a nonempty open set in V that does not contain 0. It is easy to see that U_2 is also convex, because E and U_1 are convex. Under these conditions, there is a continuous linear functional λ on V such that $\lambda(v) < 0$ for every $v \in U_2$. This follows from the argument in the previous paragraphs, applied to any translate of U_2 in V that contains 0. Equivalently, this means that

$$\lambda(x) < \lambda(y) \tag{39.9}$$

for each $x \in E$ and $y \in U_1$, since $x - y \in U_2$ and thus $\lambda(x) - \lambda(y) = \lambda(x - y) < 0$. Using this, one can check that

$$\sup_{x \in E} \lambda(x) < \lambda(v_1), \tag{39.10}$$

because U_1 is an open set in V that contains v_1 and λ is not identically 0 on V, and hence there is a $y \in U_1$ such that $\lambda(y) < \lambda(v_1)$. In particular, this implies that E is a closed set with respect to the weak topology on V, since v_1 is an arbitrary element of $V \backslash E$.

40 Convex cones

Let V be a vector space over the real numbers, and let us say that $E \subseteq V$ is a *cone* if $t\,v \in E$ for every $v \in V$ and $t \in \mathbf{R}$ with $t \geq 0$. Thus $0 \in E$ when E is a nonempty cone in V. Sometimes it is convenient to restrict one's attention to $t > 0$ in the definition of a cone, but we shall basically be concerned with closed cones here, so that we may as well include $t = 0$. A convex cone in V is a cone that is also a convex set, which is the same as a cone $E \subseteq V$ such that $v + w \in E$ for every $v, w \in E$.

If E is any subset of V, then let $C(E)$ be the cone generated by E, consisting of all vectors of the form $t\,v$ with $v \in E$ and $t \in \mathbf{R}$ such that $t \geq 0$. It is easy to see that the convex hull $\mathrm{con}(C(E))$ of $C(E)$ is a cone in V, and hence a convex cone in V. If E is convex, then one can check directly that $C(E)$ is a convex cone in V. In particular, $C(\mathrm{con}(E))$ is a convex cone for any $E \subseteq V$, which is in fact that same as $\mathrm{con}(C(E))$. Equivalently, these are both the same as the set of vectors in V of the form $\sum_{j=1}^{n} t_j\,v_j$, where $v_1, \ldots, v_n$ are finitely many elements of E, and $t_1, \ldots, t_n$ are nonnegative real numbers.

Suppose that V is a topological vector space, and let V' be the topological dual of V, as usual. If $E \subseteq V$, then let $\Gamma(E)$ be the set of continuous linear functionals λ on V such that $\lambda(v) \geq 0$ for every $v \in E$. Thus $0 \in \Gamma(E)$ for any $E \subseteq V$, and $\lambda(v) = 0$ when $\lambda \in \Gamma(E)$ and both v and $-v$ are elements of E. Note that $\Gamma(E) = V'$ when $E = \emptyset$ or $E = \{0\}$, and that $\Gamma(E) = \{0\}$ when $E = V$. It is easy to see that $\Gamma(E)$ is a convex cone in V' for every $E \subseteq V$, and that $\Gamma(E)$ is always a closed subset of V' with respect to the weak* topology. Observe also that $\Gamma(C(E)) = \Gamma(\mathrm{con}(E)) = \Gamma(E)$ for every $E \subseteq V$, and that $\Gamma(\overline{E}) = \Gamma(E)$. If V is locally convex and E is a nonempty closed convex cone in V, then it turns out that E is equal to the set of $w \in V$ such that $\lambda(w) \geq 0$ for every $\lambda \in \Gamma(E)$.

To see this, let $w \in V \backslash E$ be given, and let us show that there is a $\lambda \in \Gamma(E)$ such that $\lambda(w) < 0$ when V is locally convex and E is a nonempty closed convex cone in V. As in the previous section, there is a continuous linear functional μ on V such that $\mu(v) < \mu(w)$ for every $v \in E$ under these conditions, as a consequence of the Hahn–Banach theorem. This implies that $\mu(w) > 0$, by taking $v = 0$, and that

$$t\,\mu(v) = \mu(t\,v) < \mu(w) \tag{40.1}$$

for every nonnegative real number t, since E is a cone. It follows that that $\mu(v) \leq 0$ for every $v \in E$, so that $\lambda = -\mu$ has the required properties.

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