Some elementary aspects of Hausdorff measure and dimension

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Abstract

Basic properties of Hausdorff content, measure, and dimension of subsets of metric spaces are discussed, especially in connection with Lipschitz mappings and topological dimension.

Contents

1 Metric spaces

Let $(M, d(x,y))$ be a metric space. Thus M is a set, and $d(x,y)$ is a nonnegative real-valued function defined for $x, y \in M$ such that $d(x,y) = 0$ if and only if $x = y$,

$$d(y,x) = d(x,y) \tag{1.1}$$

for every $x, y \in M$, and

$$d(x,z) \le d(x,y) + d(y,z) \tag{1.2}$$

for every $x, y, z \in M$.

If $p \in M$ and $r > 0$, then the open ball in M with center p and radius r is defined by

$$B(p,r) = \{q \in M : d(p,q) < r\}. \tag{1.3}$$

A set $A \subseteq M$ is said to be *bounded* if it is contained in a ball, in which case its *diameter* is defined by

$$\operatorname{diam} A = \sup\{d(x,y) : x, y \in A\}. \tag{1.4}$$

This can be interpreted as being 0 when $A = \emptyset$, and it is sometimes convenient to make the convention that the diameter of an unbounded set is $+\infty$. The *closure* $\overline{A}$ of a set $A \subseteq M$ is defined to be the union of A and the set of limit points of A in M. It is easy to see that the closure of a bounded set A is also bounded, and has the same diameter as A.

A metric space M is said to be *separable* if there is a dense set $E \subseteq M$ with only finitely or countably many elements. For example, the real line $\mathbf{R}$ with the standard metric is separable, because the set $\mathbf{Q}$ of rational numbers is countable and dense in $\mathbf{R}$. Similarly, $\mathbf{R}^n$ is separable for each positive integer n with respect to the standard Euclidean metric, because $\mathbf{Q}^n$ is a countable dense set in $\mathbf{R}^n$. Note that a metric space M is separable if and only if for each $r > 0$

there is a finite or countable set $E_r \subseteq M$ which is r-dense in M in the sense that

$$M = \bigcup_{y \in E_r} B(y, r). \tag{1.5}$$

Equivalently, (1.5) says that for every $x \in M$ there is a $y \in E_r$ which satisfies $d(x, y) < r$. For if M is separable, and $E \subseteq M$ is a dense set with only finitely or countably many elements, then one can take $E_r = E$ for every $r > 0$. Conversely, if for every $r > 0$ there is an r-dense set E_r in M with only finitely or countably many elements, then

$$E = \bigcup_{n=1}^{\infty} E_{1/n} \tag{1.6}$$

is a dense set in M with only finitely or countably many elements, and hence M is separable.

A set $A \subseteq M$ is *totally bounded* if for every $r > 0$ there finitely many elements $x_1, \ldots, x_n$ of A such that

$$A \subseteq \bigcup_{i=1}^{n} B(x_i, r). \tag{1.7}$$

Totally bounded sets are bounded, and bounded subsets of $\mathbf{R}^n$ are totally bounded. Totally bounded metric spaces are automatically separable, by the remarks of the preceding paragraph. Compact sets are totally bounded, and it is well known that totally bounded closed subsets of complete metric spaces are compact.

2 Hausdorff content, 1

Let $(M, d(x, y))$ be a metric space, and let α be a positive real number. If $A \subseteq M$, then put

$$\widetilde{H}^{\alpha}_{con}(A) = \inf \left\{ \sum_{i=1}^{n} (\operatorname{diam} E_i)^{\alpha} : E_1, \ldots, E_n \subseteq M, \ A \subseteq \bigcup_{i=1}^{n} E_i \right\}. \tag{2.1}$$

More precisely, we take the infimum of the sum

$$\sum_{i=1}^{n} (\operatorname{diam} E_i)^{\alpha} \tag{2.2}$$

over all coverings of A by finitely many subsets $E_1, \ldots, E_n$ of A in M, where n depends on the covering. If any of the E_i's is unbounded, so that $\operatorname{diam} E_i = +\infty$, then (2.2) is $+\infty$ too. Otherwise, we can restrict our attention to coverings of A by finitely many bounded subsets of M when A is bounded, and set

$$\widetilde{H}^{\alpha}_{con}(A) = +\infty \tag{2.3}$$

when A is unbounded.

If $\alpha = 0$, then let us make the following conventions. First, $(\operatorname{diam} E)^0 = 0$ when $E = \emptyset$. Second, $(\operatorname{diam} E)^0 = 1$ when $E \neq \emptyset$ and E is bounded. Third, $(\operatorname{diam} E)^0 = +\infty$ when E is unbounded. In this way, we can extend the definition of $\widetilde{H}^\alpha_{con}(A)$ to $\alpha = 0$.

For every $A \subseteq M$,

$$\widetilde{H}^\alpha_{con}(A) \le (\operatorname{diam} A)^\alpha \tag{2.4}$$

This basically corresponds to the $n = 1$ case of the definition of $\widetilde{H}^\alpha_{con}(A)$, which is to say that A is covered by only one set. By allowing coverings by several sets, $\widetilde{H}^\alpha_{con}(A)$ may be reduced significantly. For instance, if A is the union of two or more subsets of M with small diameter which are relatively far from each other and $\alpha > 0$, then $\widetilde{H}^\alpha_{con}(A)$ is much smaller than $(\operatorname{diam} A)^\alpha$. Coverings by several sets is not important in the definition of $\widetilde{H}^\alpha_{con}(A)$ when $\alpha = 0$, in which event we have equality in (2.4).

Observe that $\widetilde{H}^\alpha_{con}(A)$ is monotone in A, in the sense that

$$\widetilde{H}^\alpha_{con}(A) \le \widetilde{H}^\alpha_{con}(B) \tag{2.5}$$

when $A \subseteq B \subseteq M$. This is because every covering of B is also a covering of A. In particular,

$$\widetilde{H}^\alpha_{con}(B) = 0 \tag{2.6}$$

implies that

$$\widetilde{H}^\alpha_{con}(A) = 0 \tag{2.7}$$

when $A \subseteq B$. Of course, $\widetilde{H}^\alpha_{con}(\emptyset) = 0$.

3 Closed sets

If we restrict our attention to coverings of $A \subseteq M$ by finitely many closed subsets $E_1, \ldots, E_n$ of M, then $\widetilde{H}^\alpha_{con}(A)$ would still be the same as before. For if $E_1, \ldots, E_n$ are arbitrary subsets of M such that

$$A \subseteq \bigcup_{i=1}^n E_i, \tag{3.1}$$

then their closures $\overline{E_1}, \ldots, \overline{E_n}$ are closed subsets of M which satisfy

$$A \subseteq \bigcup_{i=1}^n \overline{E_i} \tag{3.2}$$

and

$$\sum_{i=1}^n (\operatorname{diam} \overline{E_i})^\alpha = \sum_{i=1}^n (\operatorname{diam} E_i)^\alpha, \tag{3.3}$$

since $\operatorname{diam} \overline{E_i} = \operatorname{diam} E_i$ for each i. As a consequence,

$$\widetilde{H}^\alpha_{con}(\overline{A}) = \widetilde{H}^\alpha_{con}(A) \tag{3.4}$$

for any set $A \subseteq M$. Indeed, any covering of A by finitely many closed sets also covers $\overline{A}$, since the union of finitely many closed sets is closed.

4 Open sets

If $E \subseteq M$ and $r > 0$, then put

$$E(r) = \bigcup_{x \in E} B(x, r). \tag{4.1}$$

This is the same as the set of $y \in M$ for which there is an $x \in E$ such that $d(x, y) < r$. In particular, $E(r) = M$ exactly when E is r-dense in M.

By construction, $E(r)$ is an open set in M, and

$$E \subseteq E(r). \tag{4.2}$$

If E is bounded, then $E(r)$ is bounded, and

$$\operatorname{diam} E(r) \le \operatorname{diam} E + 2\, r. \tag{4.3}$$

This is easy to see, directly from the definitions.

It follows that $\widetilde{H}^\alpha_{con}(A)$ would also be the same if we restricted ourselves to coverings of A by finitely many open subsets of M. For if $E_1, \ldots, E_n$ is any collection of finitely many subsets of M that covers A, then $E_1(r), \ldots, E_n(r)$ are finitely many open subsets of M covering A for each $r > 0$, and

$$\lim_{r \to 0} \sum_{i=1}^n (\operatorname{diam} E_i(r))^\alpha = \sum_{i=1}^n (\operatorname{diam} E_i)^\alpha. \tag{4.4}$$

5 Intervals

If a, b are real numbers with $a < b$, then the open and closed intervals in the real line with endpoints a, b are defined by

$$(a, b) = \{x \in \mathbf{R} : a < x < b\} \tag{5.1}$$

and

$$[a, b] = \{x \in \mathbf{R} : a \le x \le b\}, \tag{5.2}$$

respectively. The latter also makes sense when $a = b$, and reduces to a single point. Note that

$$\operatorname{diam}\, (a, b) = \operatorname{diam}\, [a, b] = b - a. \tag{5.3}$$

If E is a bounded nonempty set in the real line, then

$$I = [\inf E, \sup E] \tag{5.4}$$

is a closed interval that contains E and has the same diameter. If E is unbounded, then one can consider $\mathbf{R}$ as an unbounded interval that contains E. Similarly, any set $E \subseteq \mathbf{R}$ is contained in an open interval whose diameter is arbitrarily close to the diameter of E.

As in the previous sections, it follows that $\widetilde{H}^{\alpha}_{con}(A)$ would be the same when $M = \mathbf{R}$ with the standard metric if we restricted our attention to coverings of A by finitely many open or closed intervals.

It is well known that

$$\widetilde{H}^{1}_{con}((a,b)) = \widetilde{H}^{1}_{con}([a,b]) = b - a. \tag{5.5}$$

By (2.4) and (2.5),

$$\widetilde{H}^{1}_{con}((a,b)) \le \widetilde{H}^{1}_{con}([a,b]) \le b - a, \tag{5.6}$$

and so one only has to show that

$$b - a \le \widetilde{H}^{1}_{con}((a,b)). \tag{5.7}$$

To see this, it suffices to check that

$$b - a \le \sum_{j=1}^{n} \operatorname{diam} I_j \tag{5.8}$$

when $I_1, \ldots, I_n$ are finitely many intervals covering (a,b).

6 Subadditivity

In any metric space M,

$$\widetilde{H}^{\alpha}_{con}(A \cup B) \le \widetilde{H}^{\alpha}_{con}(A) + \widetilde{H}^{\alpha}_{con}(B) \tag{6.1}$$

for every $A, B \subseteq M$, basically because coverings of A and B can be combined to give coverings of $A \cup B$. In particular,

$$\widetilde{H}^{\alpha}_{con}(A \cup B) = 0 \tag{6.2}$$

when $\widetilde{H}^{\alpha}_{con}(A) = \widetilde{H}^{\alpha}_{con}(B) = 0$. Similarly,

$$\widetilde{H}^{\alpha}_{con}(A \cup B) = \widetilde{H}^{\alpha}_{con}(A) \tag{6.3}$$

for every $A, B \subseteq M$ with $\widetilde{H}^{\alpha}_{con}(B) = 0$, because

$$\widetilde{H}^{\alpha}_{con}(A) \le \widetilde{H}^{\alpha}_{con}(A \cup B) \le \widetilde{H}^{\alpha}_{con}(A) + \widetilde{H}^{\alpha}_{con}(B) = \widetilde{H}^{\alpha}_{con}(A). \tag{6.4}$$

Note that $\widetilde{H}^{\alpha}_{con}(B) = 0$ when B has only finitely many elements and $\alpha > 0$, while $\widetilde{H}^{0}_{con}(B) \ge 1$ when $B \ne \emptyset$.

7 Hausdorff content, 2

Let $(M, d(x,y))$ be a metric space, and let α be a nonnegative real number. If $A \subseteq M$, then put

$$H^\alpha_{con}(A) = \inf \Big\{ \sum_i (\operatorname{diam} E_i)^\alpha : E_i \subseteq M,\ A \subseteq \bigcup_i E_i \Big\}, \tag{7.1}$$

where more precisely the infimum is taken over coverings of A by finitely or countably many subsets E_i of M. In the case of countable coverings of A,

$$\sum_i (\operatorname{diam} E_i)^\alpha \tag{7.2}$$

is interpreted as the supremum of the subsums of $(\operatorname{diam} E_i)^\alpha$ over finitely many indices i, which may be infinite. We also use the same conventions for $\alpha = 0$ as before.

Note that

$$H^\alpha_{con}(A) \le \widetilde{H}^\alpha_{con}(A), \tag{7.3}$$

since the finite coverings used in the definition of $\widetilde{H}^\alpha_{con}(A)$ are also included in the definition of $H^\alpha_{con}(A)$. In particular,

$$H^\alpha_{con}(A) \le (\operatorname{diam} A)^\alpha, \tag{7.4}$$

as in (2.4). We also have the monotonicity property

$$H^\alpha_{con}(A) \le H^\alpha_{con}(B) \tag{7.5}$$

when $A \subseteq B \subseteq M$, as in the previous case.

If we restrict our attention to coverings of A by finitely or countably many closed subsets E_i of M, then $H^\alpha_{con}(A)$ would still be the same, as in Section 3. However, this does not imply that

$$H^\alpha_{con}(\overline{A}) = H^\alpha_{con}(A), \tag{7.6}$$

since the union of infinitely many closed subsets of M may not be closed. One can also check that $H^\alpha_{con}(A)$ would be the same if we restricted our attention to coverings of A by finitely or countably many open subsets E_i of M, as in Section 4. For if $\{E_i\}_i$ is any collection of finitely or countably many subsets of M that covers A, then one can choose positive real numbers r_i such that

$$\sum_i (\operatorname{diam} E_i(r_i))^\alpha \tag{7.7}$$

is arbitrarily close to $\sum_i (\operatorname{diam} E_i)^\alpha$. If M is the real line with the standard metric, then we can use coverings of $A \subseteq \mathbf{R}$ by finitely or countably many open or closed intervals and get the same result for $H^\alpha_{con}(A)$, as in Section 5.

If K is a compact set in any metric space M, then

$$H^\alpha_{con}(K) = \widetilde{H}^\alpha_{con}(K). \tag{7.8}$$

Indeed, we can restrict our attention to coverings of K by open subsets of M, as in the previous paragraph, and then reduce to finite coverings by compactness.

8 Countable subadditivity

As before,

$$H^\alpha_{con}(A \cup B) \le H^\alpha_{con}(A) + H^\alpha_{con}(B) \tag{8.1}$$

for every $A, B \subseteq M$. Moreover,

$$H^\alpha_{con}\Big(\bigcup_{j=1}^\infty A_j\Big) \le \sum_{j=1}^\infty H^\alpha_{con}(A_j) \tag{8.2}$$

for any sequence $A_1, A_2, \ldots$ of subsets of M. Of course, the sum on the right may be $+\infty$, either because $H^\alpha_{con}(A_j) = +\infty$ for some j, or $H^\alpha_{con}(A_j) < +\infty$ for each j but the sum diverges, in which event the inequality is trivial.

To prove (8.2), we may as well suppose that $H^\alpha_{con}(A_j) < +\infty$ for each j, and that their sum converges. Let $\epsilon > 0$ be given, and for each j, let $\{E_{j,l}\}_l$ be a collection of finitely or countably many subsets of M such that

$$A_j \subseteq \bigcup_l E_{j,l} \tag{8.3}$$

and

$$\sum_l (\operatorname{diam} E_{j,l})^\alpha < H^\alpha_{con}(A_j) + 2^{-j}\,\epsilon. \tag{8.4}$$

By combining these families, we get a collection $\{E_{j,l}\}_{j,l}$ of finitely or countably many subsets of M such that

$$\bigcup_{j=1}^\infty A_j \subseteq \bigcup_{j,l} E_{j,l} \tag{8.5}$$

and

$$\sum_{j,l} (\operatorname{diam} E_{j,l})^\alpha < \sum_{j=1}^\infty (H^\alpha_{con}(A_j) + 2^{-j}\,\epsilon) = \sum_{j=1}^\infty H^\alpha_{con}(A_j) + \epsilon. \tag{8.6}$$

Thus

$$H^\alpha_{con}\Big(\bigcup_{j=1}^\infty A_j\Big) < \sum_{j=1}^\infty H^\alpha_{con}(A_j) + \epsilon, \tag{8.7}$$

which implies (8.2), since $\epsilon > 0$ is arbitrary.

In particular,

$$H^\alpha_{con}\Big(\bigcup_{j=1}^\infty A_j\Big) = 0 \tag{8.8}$$

when $H^\alpha_{con}(A_j) = 0$ for each j. It follows that

$$H^\alpha_{con}(A) = 0 \tag{8.9}$$

for every countable set A when $\alpha > 0$.

By contrast, if M is unbounded, then there are unbounded countable subsets A of M, for which

$$\widetilde{H}^{\alpha}_{con}(A) = +\infty. \tag{8.10}$$

If M is the real line with the standard metric, and A is the set of rational numbers in an interval $[a, b]$, $a < b$, then A is a bounded countable set, while

$$\widetilde{H}^{1}_{con}(A) = \widetilde{H}^{1}_{con}(\overline{A}) = \widetilde{H}^{1}_{con}([a, b]) = b - a > 0. \tag{8.11}$$

Observe that

$$\widetilde{H}^{1}_{con}(\overline{A}) = H^{1}_{con}(\overline{A}), \tag{8.12}$$

since $\overline{A} = [a, b]$ is compact, and so we get examples where

$$H^{1}_{con}(A) < H^{1}_{con}(\overline{A}), \tag{8.13}$$

as in the previous section.

9 Borel measures

Let μ be a nonnegative Borel measure on a metric space M such that

$$\mu(E) \le C\,(\operatorname{diam} E)^{\alpha} \tag{9.1}$$

for some $\alpha, C > 0$ and every Borel set E in M. For example, Lebesgue measure on $\mathbf{R}^n$ has this property with $\alpha = n$, and with $C = 1$ when $n = 1$. In this case,

$$\mu(A) \le C\, H^{\alpha}_{con}(A) \tag{9.2}$$

for every Borel set A in M. For if $\{E_i\}_i$ is any collection of finitely or countably many Borel sets in M such that

$$A \subseteq \bigcup_i E_i, \tag{9.3}$$

then

$$\mu(A) \le \sum_i \mu(E_i) \le C \sum_i (\operatorname{diam} E_i)^{\alpha}. \tag{9.4}$$

The desired estimate follows, since it suffices to consider coverings of A by open or closed subsets of M in the definition of $H^{\alpha}_{con}(A)$, which are automatically Borel sets. This is a basic technique to get lower bounds for $H^{\alpha}_{con}(A)$, while upper bounds can be obtained from explicit coverings of A. In particular, this is a common way to show that $H^{\alpha}_{con}(A) > 0$.

10 Localization

Let M be a metric space, and fix $\alpha \ge 0$, $0 < \delta \le \infty$. If $A \subseteq M$, then put

$$(10.1)\quad \widetilde{H}^\alpha_\delta(A) = \inf\Big\{ \sum_{i=1}^n (\operatorname{diam} E_i)^\alpha : E_1, \ldots, E_n \subseteq M,\ A \subseteq \bigcup_{i=1}^n E_i, \text{ and } \operatorname{diam} E_i < \delta \text{ for } i = 1, \ldots, n \Big\}$$

and

$$(10.2)\quad H^\alpha_\delta(A) = \inf\Big\{ \sum_i (\operatorname{diam} E_i)^\alpha :\ E_i \subseteq M,\ A \subseteq \bigcup_i E_i, \text{ and } \operatorname{diam} E_i < \delta \text{ for each } i \Big\}.$$

In the first case, the infimum is taken over all coverings of A by finitely many sets with diameter less than δ, while in the second case the infimum is taken over all coverings of A by finitely or countably many sets with diameter less than δ. This is interpreted as being $+\infty$ when there is no such covering of A. If A is totally bounded, then there are admissible coverings of A for $\widetilde{H}^\alpha_\delta(A)$ for every $\delta > 0$. Similarly, there are admissible coverings of A for $H^\alpha_\delta(A)$ for every $\delta > 0$ when M is separable. However, the corresponding infinite sums $\sum_i (\operatorname{diam} E_i)^\alpha$ may still diverge, so that $H^\alpha_\delta(A) = +\infty$. Of course,

$$(10.3)\quad H^\alpha_\delta(A) \le \widetilde{H}^\alpha_\delta(A),$$

since the coverings of A used in the definition of $\widetilde{H}^\alpha_\delta(A)$ are more restrictive than the coverings used in the definition of $H^\alpha_\delta(A)$.

By definition,

$$(10.4)\quad \widetilde{H}^\alpha_{con}(A) = \widetilde{H}^\alpha_\infty(A)$$

and

$$(10.5)\quad H^\alpha_{con}(A) = H^\alpha_\infty(A)$$

for every $A \subseteq M$ and $\alpha \ge 0$. Also,

$$(10.6)\quad \widetilde{H}^\alpha_\eta(A) \le \widetilde{H}^\alpha_\delta(A)$$

and

$$(10.7)\quad H^\alpha_\eta(A) \le H^\alpha_\delta(A)$$

when $0 < \delta \le \eta \le \infty$. This is because the infima in the definitions of $\widetilde{H}^\alpha_\delta(A)$, $H^\alpha_\delta(A)$ are taken over more restrictive classes of coverings of A as δ decreases, which implies that these infima are increasing as δ decreases. As a partial converse to this statement, $\widetilde{H}^\alpha_\delta(A)$ or $H^\alpha_\delta(A)$ is equal to 0 for every $\delta > 0$ as soon as it is 0 for some $\delta > 0$. For the sets E_i in the appropriate coverings of

A have to have small diameter when the corresponding sums $\sum_i (\operatorname{diam} E_i)^\alpha$ are small.

As before,

$$\widetilde{H}^\alpha_\delta(A) \le \widetilde{H}^\alpha_\delta(B) \tag{10.8}$$

and

$$H^\alpha_\delta(A) \le H^\alpha_\delta(B) \tag{10.9}$$

when $A \subseteq B$, since every admissible covering of B for $\widetilde{H}^\alpha_\delta(B)$ or $H^\alpha_\delta(B)$ is also an admissible covering of A. Moreover, $\widetilde{H}^\alpha_\delta(A)$ is finitely subadditive and $H^\alpha_\delta(A)$ is countably subadditive for every $\delta > 0$, for the same reasons as when $\delta = +\infty$. We may restrict our attention to coverings of A by open or closed sets in the definitions of $\widetilde{H}^\alpha_\delta(A)$ and $H^\alpha_\delta(A)$, or to coverings by intervals when $M = \mathbf{R}$, again for the same reasons as when $\delta = +\infty$. This implies that

$$\widetilde{H}^\alpha_\delta(\overline{A}) = \widetilde{H}^\alpha_\delta(A) \tag{10.10}$$

for every $A \subseteq M$, and that

$$H^\alpha_\delta(K) = \widetilde{H}^\alpha_\delta(K) \tag{10.11}$$

when $K \subseteq M$ is compact, as when $\delta = +\infty$.

If $A, B \subseteq M$ and

$$d(x, y) \ge \delta \tag{10.12}$$

for every $x \in A$ and $y \in B$, then

$$\widetilde{H}^\alpha_\delta(A \cup B) = \widetilde{H}^\alpha_\delta(A) + \widetilde{H}^\alpha_\delta(B) \tag{10.13}$$

and

$$H^\alpha_\delta(A \cup B) = H^\alpha_\delta(A) + H^\alpha_\delta(B). \tag{10.14}$$

For if $\{E_i\}_i$ is a covering of $A \cup B$ by sets of diameter less than δ, then each E_i will intersect at most one of A and B. This implies that there are disjoint subcollections of $\{E_i\}_i$ covering A and B, which permits one to estimate the sum of the measures of A and B by the corresponding measure of $A \cup B$.

11 Hausdorff measures

Let M be a metric space, and fix $\alpha \ge 0$. If $A \subseteq M$, then put

$$\widetilde{H}^\alpha(A) = \sup_{\delta > 0} \widetilde{H}^\alpha_\delta(A) \tag{11.1}$$

and

$$H^\alpha(A) = \sup_{\delta > 0} H^\alpha_\delta(A). \tag{11.2}$$

The latter is known as the *α-dimensional Hausdorff measure* of A. Note that $\widetilde{H}^\alpha_\delta(A)$ or $H^\alpha_\delta(A)$ may be $+\infty$ for some $\delta > 0$, or they may be finite for every δ

but unbounded, so that the supremum is $+\infty$. The supremum over $\delta > 0$ can also be considered as a limit as $\delta \to 0$, since $\widetilde{H}^\alpha_\delta(A)$ and $H^\alpha_\delta(A)$ are monotone increasing as δ decreases.

As usual,

$$H^\alpha(A) \le \widetilde{H}^\alpha(A) \tag{11.3}$$

for every $A \subseteq M$, and

$$\widetilde{H}^\alpha(A) \le \widetilde{H}^\alpha(B) \tag{11.4}$$

and

$$H^\alpha(A) \le H^\alpha(B) \tag{11.5}$$

when $A \subseteq B \subseteq M$. In addition, $\widetilde{H}^\alpha$ is finitely subadditive and H^α is countably subadditive, because of the corresponding properties of $\widetilde{H}^\alpha_\delta$ and H^α_δ. Similarly,

$$\widetilde{H}^\alpha(\overline{A}) = \widetilde{H}^\alpha(A) \tag{11.6}$$

for every $A \subseteq M$, and

$$H^\alpha(K) = \widetilde{H}^\alpha(K) \tag{11.7}$$

when K is compact. Of course,

$$\widetilde{H}^\alpha_{con}(A) \le \widetilde{H}^\alpha(A) \tag{11.8}$$

and

$$H^\alpha_{con}(A) \le H^\alpha(A). \tag{11.9}$$

Conversely,

$$\widetilde{H}^\alpha_{con}(A) = 0 \text{ implies } \widetilde{H}^\alpha(A) = 0 \tag{11.10}$$

and

$$H^\alpha_{con}(A) = 0 \text{ implies } H^\alpha(A) = 0, \tag{11.11}$$

by the analogous remarks in the previous section.

If $A, B \subseteq M$ and

$$d(x, y) \ge \eta \tag{11.12}$$

for some $\eta > 0$ and every $x \in A$, $y \in B$, then

$$\widetilde{H}^\alpha(A \cup B) = \widetilde{H}^\alpha(A) + \widetilde{H}^\alpha(B) \tag{11.13}$$

and

$$H^\alpha(A \cup B) = H^\alpha(A) + H^\alpha(B), \tag{11.14}$$

by the analogous statement in the previous section. For example, one can use this to show that $\widetilde{H}^0 = H^0$ is the same as counting measure. If $M = \mathbf{R}$, then $H^1(A)$ is the same as the Lebesgue outer measure of A. In this case, one can check that $\widetilde{H}^1_\delta(A)$ and $H^1_\delta(A)$ do not depend on δ.

If $\alpha < \beta$, then it is easy to see that

$$\widetilde{H}^\beta_\delta(A) \le \delta^{\beta-\alpha}\, \widetilde{H}^\alpha_\delta(A) \tag{11.15}$$

and

$$H^\beta_\delta(A) \le \delta^{\beta-\alpha}\, H^\alpha_\delta(A). \tag{11.16}$$

This implies that $\widetilde{H}^\beta(A) = 0$ when $\widetilde{H}^\alpha(A) < +\infty$, and that $H^\beta(A) = 0$ when $H^\alpha(A) < +\infty$. Equivalently, $\widetilde{H}^\alpha(A) = +\infty$ when $\widetilde{H}^\beta(A) > 0$, and $H^\alpha(A) = +\infty$ when $H^\beta(A) > 0$.

12 Borel sets

Let M be a metric space, and suppose that $A \subseteq M$ satisfies

$$H^\alpha(A) < +\infty \tag{12.1}$$

for some $\alpha \ge 0$. For each positive integer l, let $\{E_{j,l}\}_j$ be a collection of finitely or countably many open subsets of M such that

$$\operatorname{diam} E_{j,l} < \frac{1}{l} \tag{12.2}$$

for each j,

$$A \subseteq \bigcup_j E_{j,l}, \tag{12.3}$$

and

$$\sum_j (\operatorname{diam} E_{j,l})^\alpha < H^\alpha_{1/l}(A) + \frac{1}{l}. \tag{12.4}$$

Put

$$B = \bigcap_{l=1}^\infty \Big(\bigcup_j E_{j,l}\Big), \tag{12.5}$$

so that $A \subseteq B$ by construction. Hence $H^\alpha(A) \le H^\alpha(B)$, while

$$H^\alpha_{1/l}(B) \le \sum_j (\operatorname{diam} E_{j,l})^\alpha < H^\alpha_{1/l}(A) + \frac{1}{l} \tag{12.6}$$

for each l, because $\{E_{j,l}\}_j$ also covers B. Thus $H^\alpha(B) \le H^\alpha(A)$, and therefore

$$H^\alpha(A) = H^\alpha(B). \tag{12.7}$$

Note that $\bigcup_j E_{j,l}$ is an open set for each l, since the $E_{j,l}$'s are open sets. It follows that B is the intersection of a sequence of open sets, and hence a Borel set.

13 Borel measures, 2

Let M be a metric space, let μ be a Borel measure on M, and fix $\alpha, \delta > 0$. If

$$\mu(E) \le C\,(\operatorname{diam} E)^\alpha \tag{13.1}$$

for some $C > 0$ and every Borel set $E \subseteq M$ with $\operatorname{diam} E < \delta$, then

$$\mu(A) \le C\, H^\alpha_\delta(A), \tag{13.2}$$

for every Borel set $A \subseteq M$, by the same argument as in Section 9. Actually, it suffices to restrict our attention to open sets E, since we can use coverings by open sets in the definition of H^α_δ. However, one can also check that (13.1) automatically holds for arbitrary Borel sets when it holds for open sets and μ is outer regular. It suffices as well to consider only sets E such that

$$A \cap E \ne \emptyset. \tag{13.3}$$

This is because sets disjoint from A can be dropped from any covering of A without affecting the estimates. Put

$$X(\alpha,\delta,C) = \{x \in M : \mu(E) \le C\,(\operatorname{diam} E)^\alpha \text{ for every open set } E \subseteq M \text{ with } x \in E \text{ and } \operatorname{diam} E < \delta\}. \tag{13.4}$$

The preceding remarks imply that (13.2) holds when $A \subseteq X(\alpha,\delta,C)$. An advantage of working with open sets E is that $X(\alpha,\delta,C)$ is automatically a closed set in M, because its complement is an open set.

14 A refinement

Let us continue with the same notations as in the previous section. Observe that

$$X(\alpha,\eta,C) \subseteq X(\alpha,\delta,C) \tag{14.1}$$

when $\delta \le \eta$. Put

$$X(\alpha,C) = \bigcup_{\delta>0} X(\alpha,\delta,C), \tag{14.2}$$

which is the same as

$$X(\alpha,C) = \bigcup_{l=1}^\infty X(\alpha,1/l,C), \tag{14.3}$$

by monotonicity in δ. Thus $X(\alpha,C)$ is a Borel set, since it is the union of a sequence of closed sets. We would like to show that

$$\mu(A) \le C\, H^\alpha(A) \tag{14.4}$$

when A is a Borel set such that $A \subseteq X(\alpha,C)$. Consider

$$A_l = A \cap X(\alpha,1/l,C). \tag{14.5}$$

This is a Borel set for each l, and

$$\mu(A_l) \le C\, H^\alpha_{1/l}(A_l) \le C\, H^\alpha(A_l), \tag{14.6}$$

as in the previous section. Hence

$$\mu(A_l) \le C\, H^\alpha(A) \tag{14.7}$$

for each l, which implies (14.4), because $A = \bigcup_{l=1}^\infty A_l$ and $A_l \subseteq A_{l+1}$ for each l.

15 Hausdorff dimension

The *Hausdorff dimension* $\dim_H A$ of a set A in a metric space M can normally be defined as the supremum of the set of $\alpha \ge 0$ such that $H^\alpha(A) = +\infty$, or as the infimum of the set of $\beta \ge 0$ such that $H^\beta(A) = 0$. These two quantities are the same when they are both defined, because $H^\beta(A) = 0$ when $H^\alpha(A) < +\infty$ and $\beta > \alpha$, as in Section 11. If A has only finitely many elements, then $H^0(A) < +\infty$ and $H^\alpha(A) = 0$ for every $\alpha > 0$, and $\dim_H A = 0$. Similarly, $\dim_H A = +\infty$ when $H^\alpha(A) = +\infty$ for every $\alpha \ge 0$.

If $A \subseteq B \subseteq M$, then

$$\dim_H A \le \dim_H B. \tag{15.1}$$

If $A_1, A_2, \ldots$ is a sequence of subsets of M, then

$$\dim_H\Big(\bigcup_{j=1}^\infty A_j\Big) = \sup_{j\ge 1} \dim_H A_j. \tag{15.2}$$

Indeed, (15.1) implies that

$$\dim_H A_i \le \dim_H\Big(\bigcup_{j=1}^\infty A_j\Big) \tag{15.3}$$

for each i, and the opposite inequality follows from countable subadditivity.

Note that $\dim_H A = \alpha$ when

$$0 < H^\alpha(A) < +\infty. \tag{15.4}$$

For example, an interval of positive length in the real line has Hausdorff dimension 1, and the Hausdorff dimension of a ball or a cube in $\mathbf{R}^n$ is n.

16 Separating sets

Let A be a closed set in $\mathbf{R}^n$, where the latter is equipped with the standard Euclidean metric. If $\mathbf{R}^n \backslash A$ is not connected, then

$$H^{n-1}(A) > 0. \tag{16.1}$$

To see this, let p, q be elements of different connected components of $\mathbf{R}^n \backslash A$. Thus $p \neq q$, and there is a unique line L passing through them. Let P be the hyperplane in $\mathbf{R}^n$ passing through p and perpindicular to L, and let π be the orthogonal projection of $\mathbf{R}^n$ onto P. A key point now is that

$$\pi(A) \text{ contains a neighborhood of } p \text{ in } P. \tag{16.2}$$

For if $z \in P \backslash \pi(A)$, then the line $L(z)$ passing through z and parallel to L is disjoint from A. If z is sufficiently close to p, then $L(z)$ intersects each of the complementary components of A containing p and q, a contradiction. It follows that

$$H^{n-1}(\pi(A)) > 0. \tag{16.3}$$

The remaining point is that

$$H^{n-1}(A) \geq H^{n-1}(\pi(A)), \tag{16.4}$$

basically because coverings of A can be projected onto coverings of $\pi(A)$ without increasing the diameters of the sets in the coverings.

17 Lipschitz mappings

Let $(M_1, d_1(x, y))$ and $(M_2, d_2(u, v))$ be metric spaces. A mapping $f : M_1 \to M_2$ is said to be *Lipschitz* if there is a $k \geq 0$ such that

$$d_2(f(x), f(y)) \leq k \, d_1(x, y) \tag{17.1}$$

for every $x, y \in M_1$. We may also say that f is k-Lipschitz in this case, to be more precise. Thus a mapping is 0-Lipschitz if and only if it is constant. If $M_1 = M_2$ with the same metric, then the identity mapping $f(x) = x$ is 1-Lipschitz. Orthogonal projections of $\mathbf{R}^n$ onto affine subspaces are also 1-Lipschitz. Another class of examples will be given in the next section.

Suppose that $f : M_1 \to M_2$ is k-Lipschitz. If $E \subseteq M_1$ is bounded, then $f(E)$ is bounded in M_2, and

$$\operatorname{diam} f(E) \leq k \operatorname{diam} E. \tag{17.2}$$

Using this, one can check that

$$H^\alpha_{k\delta}(f(A)) \leq k^\alpha \, H^\alpha_\delta(A) \tag{17.3}$$

for every $A \subseteq M_1$, $\alpha \geq 0$, and $0 < \delta \leq \infty$, and hence

$$H^\alpha(f(A)) \leq k^\alpha \, H^\alpha(A). \tag{17.4}$$

In particular, $\dim_H f(A) \leq \dim_H A$.

18 Connected sets

Let $(M, d(x, y))$ be a metric space. For each $p \in M$,

$$f_p(x) = d(p, x) \tag{18.1}$$

is 1-Lipschitz as a mapping from M into $\mathbf{R}$ with the standard metric. This can be verified using the triangle inequality.

If $A \subseteq M$ is connected, then

$$\operatorname{diam} A \le H^1(A). \tag{18.2}$$

To see this, we can use the remarks in the previous section to get that

$$H^1(f_p(A)) \le H^1(A) \tag{18.3}$$

for every $p \in M$. The connectedness of A and continuity of f_p imply that $f_p(A)$ is basically an interval in $\mathbf{R}$, whose 1-dimensional Hausdorff measure is the same as its length, which is the same as its diameter, so that

$$\operatorname{diam} f_p(A) \le H^1(A). \tag{18.4}$$

This implies (18.2), because the diameter of A is the same as the supremum of the diameter of $f_p(A)$ over $p \in A$.

19 Bilipschitz enbeddings

Let $(M_1, d_1(x, y))$, $(M_2, d_2(u, v))$ be metric spaces again. A mapping f from M_1 into M_2 is said to be *bilipschitz* if there is a $k \ge 1$ such that

$$k^{-1}\, d_1(x, y) \le d_2(f(x), f(y)) \le k\, d_1(x, y) \tag{19.1}$$

for every $x, y \in M_1$. Equivalently, $f : M_1 \to M_2$ is k-bilipschitz if f is k-Lipschitz, f is one-to-one, and the inverse mapping is k-Lipschitz on $f(M_1)$. In this case,

$$k^{-\alpha}\, H^\alpha(A) \le H^\alpha(f(A)) \le k^\alpha\, H^\alpha(A) \tag{19.2}$$

for every $A \subseteq M_1$ and $\alpha \ge 0$, and in particular

$$\dim_H f(A) = \dim_H A \tag{19.3}$$

for every $A \subseteq M$.

By contrast, Hausdorff dimensions can be changed by ordinary homeomorphisms. For instance, there are topological Cantor sets in the real line with any Hausdorff dimension α, $0 \le \alpha \le 1$. There are even Cantor sets with positive Lebesgue measure, and hence positive one-dimensional Hausdorff measure.

20 Ultrametric spaces

Let $(M, d(x,y))$ be a metric space. We say that $d(x,y)$ is an *ultrametric* on M if

$$d(x,z) \le \max(d(x,y), d(y,z)) \tag{20.1}$$

for every $x, y, z \in M$. Of course, this is stronger than the usual triangle inequality.

A basic class of examples of ultrametric spaces is given by sequence spaces. Let L be a nonempty set, and let X be the set of sequences $x = \{x_j\}_{j=1}^{\infty}$ with $x_j \in L$ for each j. Also let ρ be a positive real number, $\rho < 1$. Let $d_\rho(x,y)$ be defined for $x, y \in X$ by $d_\rho(x,y) = 0$ when $x = y$, and

$$d_\rho(x,y) = \rho^l \tag{20.2}$$

where l is the largest nonnegative integer such that $x_j = y_j$ for $j \le l$ otherwise. If $x_1 \ne y_1$, then $l = 0$, and $d_\rho(x,y) = 1$. It is not difficult to check that $d_\rho(x,y)$ satisfies the ultrametric version of the triangle inequality. Basically, this amounts to the statement that if $x, y, z \in X$, and $x_j = y_j$, $y_j = z_j$ when $j \le l$, then $x_j = z_j$ when $j \le l$. The same argument works for $\rho = 1$, in which case $d_\rho(x,y)$ reduces to the discrete metric on X. The topology on X determined by $d_\rho(x,y)$ is the same as the product topology when $\rho < 1$, where X is considered as the product of infinitely many copies of L, and L is equipped with the discrete topology. In particular, X is compact when L has only finitely many elements.

If L is a finite set with at least two elements, then X is a topological Cantor set, which is to say that it is homeomorphic to the usual middle-thirds Cantor set. If L has exactly two elements and $\rho = 1/3$, then X is bilipschitz equivalent to the middle-thirds Cantor set.

21 Closed balls

Let $(M, d(x,y))$ be a metric space. If $p \in M$ and $r \ge 0$, then the closed ball in M with center p and radius r is defined by

$$\overline{B}(p,r) = \{x \in M : d(p,x) \le r\}. \tag{21.1}$$

This is a closed and bounded set in M, with

$$\operatorname{diam} \overline{B}(p,r) \le 2\,r. \tag{21.2}$$

Suppose now that $d(x,y)$ is an ultrametric on M. In this case, $\overline{B}(p,r)$ is actually an open set in M too, because

$$B(q,r) \subseteq \overline{B}(q,r) \subseteq \overline{B}(p,r) \tag{21.3}$$

for every $q \in \overline{B}(p,r)$. Similarly, open balls in M are closed sets. Moreover,

$$\operatorname{diam} \overline{B}(p,r) \le r. \tag{21.4}$$

Let us say that $\mathcal{C}$ is a *cell* in an ultrametric space M if it is a closed ball, i.e., if it can be represented as $\overline{B}(p, r)$ for some $p \in M$ and $r \geq 0$. In some circumstances, one may prefer to use only closed balls of positive radius. If E is a bounded set in M, $p \in E$, and

$$\mathcal{C} = \overline{B}(p, \operatorname{diam} E), \tag{21.5}$$

then $E \subseteq \mathcal{C}$ and

$$\operatorname{diam} \mathcal{C} = \operatorname{diam} E. \tag{21.6}$$

This permits one to use coverings by cells in the definition of Hausdorff measure in an ultrametric space.

If $\mathcal{C}$ is a cell in an ultrametric space M, then

$$\mathcal{C} = \overline{B}(p, \operatorname{diam} \mathcal{C}) \tag{21.7}$$

for every $p \in \mathcal{C}$. If $\mathcal{C}_1$, $\mathcal{C}_2$ are cells in M such that

$$\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset, \tag{21.8}$$

then

$$\mathcal{C}_1 \subseteq \mathcal{C}_2 \quad \text{or} \quad \mathcal{C}_2 \subseteq \mathcal{C}_1. \tag{21.9}$$

This follows by representing $\mathcal{C}_1$, $\mathcal{C}_2$ as closed balls centered at the same point p.

22 Sequence spaces

Let L be a finite set with $n \geq 2$ elements, let X be the space of sequences with entries in L, and let $d_\rho(x, y)$ be the ultrametric on X associated to some $\rho \in (0, 1)$ as in Section 20. If α is the positive real number defined by

$$n\, \rho^\alpha = 1, \tag{22.1}$$

then

$$H^\alpha(X) = H^\alpha_{con}(X) = 1. \tag{22.2}$$

To see this, observe first that

$$H^\alpha(X) \leq 1, \tag{22.3}$$

since X can be covered by n^l cells of diameter ρ^l for each l. Similarly, every cell in X of diameter ρ^j can be covered by n^{l-j} cells of diameter ρ^l when $j \leq l$. Using this fact, one can replace any covering of X by finitely many cells $\mathcal{C}_1, \ldots, \mathcal{C}_k$ with a covering by subcells $\mathcal{C}'_1, \ldots, \mathcal{C}'_r$ in such a way that

$$\operatorname{diam} \mathcal{C}'_i = \rho^l \tag{22.4}$$

for some l and all $i = 1, \ldots, r$, and

$$\sum_{h=1}^{k} (\operatorname{diam} \mathcal{C}_h)^\alpha = \sum_{i=1}^{r} (\operatorname{diam} \mathcal{C}'_i)^\alpha. \tag{22.5}$$

In order for X to be covered by a collection of cells of the same diameter ρ^l, it is necessary for all n^l of these cells to be used. This implies that

$$\sum_{h=1}^{k} (\operatorname{diam} \mathcal{C}_h)^\alpha = \sum_{i=1}^{r} (\operatorname{diam} \mathcal{C}'_i)^\alpha \ge 1, \tag{22.6}$$

and hence

$$H^\alpha(X) \ge H^\alpha_{con}(X) \ge 1. \tag{22.7}$$

23 Snowflake metrics

If $a, b \ge 0$ and $0 < t \le 1$, then

$$(a+b)^t \le a^t + b^t. \tag{23.1}$$

Indeed,

$$\max(a,b) \le (a^t + b^t)^{1/t}, \tag{23.2}$$

which implies that

$$\begin{aligned} a+b &\le \max(a,b)^{1-t}\,(a^t+b^t) \\ &\le (a^t+b^t)^{(1-t)/t}\,(a^t+b^t) = (a^t+b^t)^{1/t}. \end{aligned} \tag{23.3}$$

Let $(M, d(x,y))$ be a metric space. It follows from (23.1) that $d(x,y)^t$ is also a metric on M for each $t \in (0,1)$. If $d(x,y)$ is an ultrametric on M, then it is easy to see that $d(x,y)^t$ is an ultrametric on M for every $t > 0$. It is easy to keep track of the affect of this change in the metric on diameters of subsets of M and so on as well. In particular, the α-dimensional Hausdorff measure on M with respect to the initial metric $d(x,y)$ is the same as Hausdorff measure of dimension α/t with respect to the new metric $d(x,y)^t$ for each $\alpha \ge 0$.

24 Topological dimension 0

As in [36], a separable metric space M is said to have *topological dimension* 0 if for every $p \in M$ and $r > 0$ there is an open set U in M such that $p \in U \subseteq B(p,r)$ and $\partial U = \emptyset$. Equivalently, M has topological dimension 0 if for every $p \in M$ and open set V in M with $p \in V$ there is an open set U in M such that $p \in U \subseteq V$ and $\partial U = \emptyset$, so that this condition depends only on the topology on M. If the metric $d(x,y)$ on M is an ultrametric, then every open ball in M is a closed set, and M has topological dimension 0. For any metric $d(x,y)$ on M,

$$H^1(M) = 0 \tag{24.1}$$

implies that M has topological dimension 0. Remember that

$$f_p(x) = d(p,x) \tag{24.2}$$

is a 1-Lipschitz function for each $p \in M$, and hence

$$H^1(f_p(M)) \le H^1(M) = 0. \tag{24.3}$$

Here $H^1(f_p(M))$ is the 1-dimensional Hausdorff measure of $f_p(M)$ as a set in the real line, with the standard metric. Thus, for each $p \in M$,

$$\{x \in M : d(p,x) = r\} = \emptyset \tag{24.4}$$

for almost every r with respect to Lebesgue measure on $\mathbf{R}$, which implies that M has topological dimension 0.

25 Distance functions

Let $(M, d(x,y))$ be a metric space. If $A \subseteq M$, $A \ne \emptyset$, and $x \in M$, then put

$$\mathrm{dist}(x, A) = \inf\{d(x,a) : a \in A\}. \tag{25.1}$$

Thus $\mathrm{dist}(x, A) = 0$ if and only if $x \in \overline{A}$. Also, the distance from any $x \in M$ to A is the same as the distance from x to the closure of A. It is easy to check that

$$\mathrm{dist}(x, A) \le \mathrm{dist}(y, A) + d(x,y) \tag{25.2}$$

for every $x, y \in M$, so that $\mathrm{dist}(x, A)$ is 1-Lipschitz in x. If $d(x,y)$ is an ultrametric on M, then $\mathrm{dist}(x, M)$ satisfies the "ultra-Lipschitz" property

$$\mathrm{dist}(x, A) \le \max(\mathrm{dist}(y, A), d(x,y)). \tag{25.3}$$

In particular,

$$\mathrm{dist}(x, A) = \mathrm{dist}(y, A) \tag{25.4}$$

when $d(x,y) < \mathrm{dist}(x, A)$.

The continuity of $\mathrm{dist}(x, A)$ implies that

$$U_r = \{x \in M : \mathrm{dist}(x, A) < r\} \tag{25.5}$$

is an open set in M for each $r > 0$, and

$$L_r = \{x \in M : \mathrm{dist}(x, A) \le r\} \tag{25.6}$$

is a closed set. If $d(x,y)$ is an ultrametric, then U_r is a closed set for each $r > 0$, and L_r is an open set. For any metric $d(x,y)$, $H^1(M) = 0$ implies that

$$\{x \in M : \mathrm{dist}(x, A) = r\} = \emptyset \tag{25.7}$$

for almost every $r > 0$, so that $U_r = L_r$ for almost every r in this case.

26 Disjoint closed sets

Let $(M, d(x,y))$ be a metric space, and let A, B be disjoint nonempty closed subsets of M. Consider the function

$$\phi(x) = \frac{\mathrm{dist}(x,A)}{\mathrm{dist}(x,A) + \mathrm{dist}(x,B)}. \tag{26.1}$$

As usual, this is a continuous real-valued function on M such that $\phi(x) = 0$ when $x \in A$, $\phi(x) = 1$ when $x \in B$, and $0 \le \phi(x) \le 1$ for every $x \in M$. For each $\epsilon > 0$, ϕ is also Lipschitz on the set

$$\Lambda_\epsilon = \{x \in M : \mathrm{dist}(x,A) + \mathrm{dist}(x,B) \ge \epsilon\}. \tag{26.2}$$

In particular, ϕ is locally Lipschitz, in the sense that ϕ is Lipschitz on a neighborhood of any point in M. If the distance between A and B is positive, then ϕ is Lipschitz on all of M. Of course, this holds automatically when A or B is compact. If $d(x,y)$ is an ultrametric on M, then $\mathrm{dist}(x,A)$ is locally constant on $M \backslash A$, $\mathrm{dist}(x,B)$ is locally constant on $M \backslash B$, and hence ϕ is locally constant on $M \backslash (A \cup B)$.

For any metric $d(x,y)$ on M, if $H^1(M) = 0$, then

$$H^1(\phi(\Lambda_\epsilon)) = 0, \tag{26.3}$$

for each $\epsilon > 0$, since ϕ is Lipschitz on Λ_ϵ. This implies that

$$H^1(\phi(M)) = 0, \tag{26.4}$$

since $M = \bigcup_{n=1}^\infty \Lambda_{1/n}$. Thus $\phi^{-1}(r) = \emptyset$ for almost every $r \in (0,1)$.

27 Topological dimension 0, 2

Let M be a separable metric space. It is easy to see that M has topological dimension 0 if and only if for each $p \in M$ and closed set B in M with $p \notin B$ there is an open set $U \subseteq M$ such that

$$p \in U,\ U \cap B = \emptyset, \text{ and } \partial U = \emptyset. \tag{27.1}$$

Let us say that M has property P if for every pair of distinct elements p, q of M there is an open set U in M such that

$$p \in U,\ q \notin U, \text{ and } \partial U = \emptyset. \tag{27.2}$$

Similarly, we say that M has property Q if for every pair A, B of disjoint closed subsets of M there is an open set U in M such that

$$A \subseteq U,\ U \cap B = \emptyset, \text{ and } \partial U = \emptyset. \tag{27.3}$$

Thus property Q implies that M has topological dimension 0, which implies property P, which implies that M is totally disconnected in the sense that it has no connected subsets with at least two elements. Note that properties P and Q are automatically symmetric in p, q and A, B, respectively, since their roles in the definitions can be interchanged. The conclusions of properties P and Q are also symmetric in p, q and A, B, because $V = M \backslash U$ is an open set in M with $\partial V = \emptyset$ when U is an open set in M with $\partial U = \emptyset$.

Suppose that M satisfies property P, $p \in M$, B is a compact set in M, and $p \notin B$. For each $q \in B$, let $V(q)$ be an open set in M such that $q \in V(q)$, $p \notin V(q)$, and $\partial V(q) = \emptyset$. By compactness, there are finitely many elements $q_1, \ldots, q_l$ of B such that

$$B \subseteq V(q_1) \cup V(q_2) \cup \cdots \cup V(q_l). \tag{27.4}$$

Hence $V = \bigcup_{i=1}^{l} V(q_i)$ is an open set in M such that $B \subseteq V$, $p \notin V$, and $\partial V = \emptyset$. Equivalently, $U = M \backslash V$ is an open set in M such that $p \in U$, $B \subseteq M \backslash U$, and $\partial U = \emptyset$. If A, B are disjoint compact subsets of M, then one can apply this to each $p \in A$ to get an open set $U(p)$ in M such that $p \in U(p)$, $B \subseteq M \backslash U(p)$, and $\partial U(p) = \emptyset$. By compactness, there are finitely many elements $p_1, \ldots, p_r$ of A such that

$$A \subseteq U(p_1) \cup U(p_2) \cup \cdots \cup U(p_r). \tag{27.5}$$

Thus $W = \bigcup_{j=1}^{r} U(p_j)$ is an open set, $A \subseteq W$, $B \subseteq M \backslash W$, and $\partial W = \emptyset$. This shows that property P implies the analogues of the reformulation of topological dimension 0 in the previous paragraph and property Q for compact subsets of M instead of closed subsets of M. In particular, property P implies that M has topological dimension 0 and satisfies property Q when M is compact.

The same type of covering argument shows that M satisfies the analogue of property Q for compact sets A in M when M has topological dimension 0. An important theorem states that any separable metric space of topological dimension 0 satisfies property Q. The proof of this will be given in Section 29.

Let A, B be disjoint nonempty closed subsets of a metric space M, and let $\phi(x)$ be as in the previous section. Thus

$$U_r = \{x \in M : \phi(x) < r\} \tag{27.6}$$

is an open set in M such that

$$A \subseteq U_r \text{ and } \overline{U_r} \cap B = \emptyset \tag{27.7}$$

for every $r \in (0, 1)$. If the metric $d(x, y)$ on M is an ultrametric, then $\partial U = \emptyset$ for each $r \in (0, 1)$, because ϕ is locally constant on $M \backslash (A \cup B)$. For any metric on M,

$$\partial U_r \subseteq \{x \in M : \phi(x) = r\}. \tag{27.8}$$

If $H^1(M) = 0$, then it follows that $\partial U_r = \emptyset$ for almost every $r \in (0, 1)$, so that M has property Q.

28 Strong normality

Remember that two subsets A, B of a topological space X are said to be *separated* if

$$\overline{A} \cap B = A \cap \overline{B} = \emptyset. \tag{28.1}$$

A strong version of normality asks that for any pair of separated subsets A, B of X there are disjoint open subsets U, V of X such that $A \subseteq U$ and $B \subseteq V$. This implies that every $Y \subseteq X$ has the same property with respect to the induced topology, because $A, B \subseteq Y$ are separated relative to Y if and only if they are separated relative to X.

If the topology on X is defined by a metric, then X satisfies this stronger version of normality. To see this, let A, B be separated subsets of X. By hypothesis, each $a \in A$ is not an element of the closure of B, and so there is an $r(a) > 0$ such that

$$B(a, r(a)) \cap B = \emptyset. \tag{28.2}$$

Similarly, for every $b \in B$ there is a $t(b) > 0$ such that

$$B(b, t(b)) \cap A = \emptyset. \tag{28.3}$$

Consider

$$U = \bigcup_{a \in A} B(a, r(a)/2), \ V = \bigcup_{b \in B} B(b, t(b)/2). \tag{28.4}$$

Thus U and V are open subsets of X such that $A \subseteq U$ and $B \subseteq V$. If $U \cap V \neq \emptyset$, then there are $a \in A$ and $b \in B$ such that

$$B(a, r(a)/2) \cap B(b, t(b)/2) \neq \emptyset. \tag{28.5}$$

This would imply that the distance from a to b is less than the average of $r(a)$ and $t(b)$, by the triangle inequality. However, the distance from a to b is greater than or equal to both $r(a)$ and $t(b)$ by construction, and so we may conclude that U and V are disjoint. Note that one can use balls of radius $r(a)$ and $t(b)$ in (28.4) when the metric on X is an ultrametric.

A classical theorem asserts that a regular topological space X with a countable base for its topology satisfies this strong version of normality. Indeed, let A, B be separated subsets of X again. Regularity implies that each $a \in A$ has a neighborhood $U(a)$ such that

$$\overline{U(a)} \cap B = \emptyset. \tag{28.6}$$

Similarly, each $b \in B$ has a neighborhood $V(b)$ such that

$$\overline{V(b)} \cap A = \emptyset. \tag{28.7}$$

Because X has a countable base for its topology, there are sequences $\{U_i\}_{i=1}^\infty$ and $\{V_j\}_{j=1}^\infty$ of open subsets of X such that

$$A \subseteq \bigcup_{i=1}^\infty U_i, \ B \subseteq \bigcup_{j=1}^\infty V_j \tag{28.8}$$

and

$$\overline{U_i} \cap B = \overline{V_j} \cap A = \emptyset \tag{28.9}$$

for every i, j. For each $l, n \geq 1$, put

$$\widetilde{U}_l = U_l \backslash \Big(\bigcup_{j=1}^{l} \overline{V_j} \Big), \ \widetilde{V}_n = V_n \backslash \Big(\bigcup_{i=1}^{n} \overline{U_i} \Big) \tag{28.10}$$

Thus $\widetilde{U}_l$, $\widetilde{V}_n$ are open sets for every $l, n \geq 1$, and

$$\widetilde{U}_l \cap \widetilde{V}_n = \emptyset. \tag{28.11}$$

This implies that

$$U = \bigcup_{l=1}^{\infty} \widetilde{U}_l, \ V = \bigcup_{n=1}^{\infty} \widetilde{V}_n \tag{28.12}$$

are disjoint open sets, and it is easy to check that $A \subseteq U$, $B \subseteq V$, as desired. Of course, Urysohn's famous metrization theorem implies that a regular topological space with a countable base for its topology is metrizable.

29 Topological dimension 0, 3

Let M be a separable metric space of topological dimension 0, let A, B be disjoint closed subsets of M, and let us show that there is an open and closed set U in M such that $A \subseteq U$ and $B \subseteq M \backslash U$, as in Section 27. For each $p \in M$, there is an open and closed set $W(p)$ in M such that $p \in W(p)$ and

$$W(p) \cap A = \emptyset \text{ or } W(p) \cap B = \emptyset. \tag{29.1}$$

Because M has a countable base for its topology, it follows that there is a sequence $W_1, W_2, \ldots$ of open and closed subsets of M such that

$$\bigcup_{i=1}^{\infty} W_i = M, \tag{29.2}$$

and

$$W_i \cap A = \emptyset \text{ or } W_i \cap B = \emptyset \tag{29.3}$$

for each i. Put $\widetilde{W}_1 = W_1$ and

$$\widetilde{W}_l = W_l \backslash \Big(\bigcup_{i=1}^{l-1} W_i \Big) \tag{29.4}$$

when $l \geq 2$. Thus $\widetilde{W}_l \subseteq W_l$ and $\widetilde{W}_l$ is both open and closed for each l, and

$$\bigcup_{l=1}^{\infty} \widetilde{W}_l = \bigcup_{i=1}^{\infty} W_i = M. \tag{29.5}$$

Consider

$$U = \bigcup \{\widetilde{W}_l : \widetilde{W}_l \cap A \neq \emptyset\} \tag{29.6}$$

and

$$V = \bigcup \{\widetilde{W}_l : \widetilde{W}_l \cap A = \emptyset\} = M \backslash U. \tag{29.7}$$

These are obviously open sets, since they are unions of open sets. Hence they are closed sets as well, because they are complements of each other. Each $\widetilde{W}_l$ can intersect at most one of A and B, which implies that $A \subseteq U$ and $B \subseteq V$, as desired.

30 Subsets

Let M be a separable metric space. A set $X \subseteq M$ is considered to have topological dimension 0 if it has topological dimension 0 as a metric space itself, with respect to the restriction of the metric on M to X. Actually, it is sometimes convenient to let the topological dimension of the empty set be -1, but this is not important for the moment. It is easy to see that $X \subseteq M$ has topological dimension 0 when M has topological dimension 0. Hence $X \subseteq Y \subseteq M$ has topological dimension 0 when Y has topological dimension 0.

Consider the case of the real line with the standard metric. It is easy to see that $X \subseteq \mathbf{R}$ has topological dimension 0 if and only if X is totally disconnected, which is the same as saying that X has at least two elements but does not contain any interval of positive length in this case. For example, the set $\mathbf{Q}$ of rational numbers has topological dimension 0, as does the set $\mathbf{R} \backslash \mathbf{Q}$ of irrational numbers. This shows that the union of two sets with topological dimension 0 may not have topological dimension 0, since the real line is connected and therefore does not have topological dimension 0.

Let M be a separable metric space again, and suppose that $X \subseteq M$ has topological dimension 0 and $M = X \cup \{p\}$ for some $p \in M \backslash X$. Thus X is an open set in M, and we would like to show that M also has topological dimension 0 in these circumstances. It is easy to see that every element of X has arbitrarily small open neighborhoods with empty boundary in M, because of the corresponding property of X. The main point is to show that the analogous statement holds at p. Let $r > 0$ be given, and consider

$$A = \overline{B}(p, r/2), \ B = M \backslash B(p, r). \tag{30.1}$$

These are disjoint closed subsets of M, and $B \subseteq X$. Hence $A \backslash \{p\}$, B are disjoint relatively closed subsets of X. The theorem in the previous section may be applied in X to get disjoint open sets U, V in X and therefore in M such that $A \backslash \{p\} \subseteq U$, $B \subseteq V$, and $U \cup V = X$. If $U_1 = U \cup \{p\}$, then U_1 is an open set in M, because $B(p, r/2) \subseteq U_1$ by construction. Moreover, $U_1 \subseteq B(p, r)$ because $U_1 \cap B = \emptyset$, and U_1 is closed in M, since $M \backslash U_1 = V$ is open.

It follows that the union of a set with topological dimension 0 and a finite set also has topological dimension 0. The analogous statement for countable sets does not work, as in the example of the sets of rational and irrational numbers in the real line.

31 Closed subsets

Let M be a separable metric space, and let $X_1 \subseteq M$ be a closed set with topological dimension 0. If A, B are disjoint closed subsets of M, then $A \cap X_1$, $B \cap X_1$ are disjoint closed subsets of X_1. The theorem in Section 29 implies that there are disjoint closed subsets $\widetilde{A}$, $\widetilde{B}$ of X_1 such that

$$A \cap X_1 \subseteq \widetilde{A}, \ B \cap X_1 \subseteq \widetilde{B}, \text{ and } \widetilde{A} \cup \widetilde{B} = X_1. \tag{31.1}$$

Thus $A_1 = A \cup \widetilde{A}$ and $B_1 = B \cup \widetilde{B}$ are disjoint closed subsets of M such that

$$A \subseteq A_1, \ B \subseteq B_1, \text{ and } X_1 \subseteq A_1 \cup B_1. \tag{31.2}$$

If X_2 is another closed set in M with topological dimension 0, then we can apply the same procedure a second time to get disjoint closed subsets A_2, B_2 of M such that

$$A \subseteq A_1 \subseteq A_2, \ B \subseteq B_1 \subseteq B_2, \text{ and } X_1 \cup X_2 \subseteq A_2 \cup B_2. \tag{31.3}$$

If $M = X_1 \cup X_2$, then $M = A_2 \cup B_2$, so that A_2, B_2 are open sets as well. This shows that the union of two closed sets with topological dimension 0 also has topological dimension 0.

Suppose now that $X_1, X_2, \ldots$ is a sequence of closed subsets of a separable metric space M, where each X_i has topological dimension 0. If $M = \bigcup_{i=1}^\infty X_i$, then a fundamental theorem states that M also has topological dimension 0. To prove this, it is not quite sufficient to repeat the process and take the union of the resulting A_i's and B_i's, because it is not clear that these will be closed sets. Instead, one can apply normality after the first step to get open subsets U_1, V_1 of M such that

$$A_1 \subseteq U_1, \ B_1 \subseteq V_1, \text{ and } \overline{U_1} \cap \overline{V_1} = \emptyset. \tag{31.4}$$

Proceeding as before, we get disjoint closed subsets $\widehat{A}_2$, $\widehat{B}_2$ of M such that

$$\overline{U_1} \subseteq \widehat{A}_2, \ \overline{V_1} \subseteq \widehat{B}_2, \text{ and } X_2 \subseteq \widehat{A}_2 \cup \widehat{B}_2. \tag{31.5}$$

Of course,

$$X_1 \subseteq A_1 \cup B_1 \subseteq U_1 \cup V_1 \subseteq \widehat{A}_2 \cup \widehat{B}_2. \tag{31.6}$$

Repeating the process, we get increasing sequences $\{U_i\}_{i=1}^\infty$, $\{V_i\}_{i=1}^\infty$ of open subsets of M, and we put $U = \bigcup_{i=1}^\infty U_i$, $V = \bigcup_{i=1}^\infty V_i$. By construction,

$$A \subseteq U, \ B \subseteq V, \ U \cap V = \emptyset, \text{ and } U \cup V = M. \tag{31.7}$$

Suppose instead that $M = X_1 \cup X_2$, where X_1, X_2 have topological dimension 0 and X_1 is closed. We may as well take $X_2 = M \backslash X_1$, because a subset of a set with topological dimension 0 also has topological dimension 0, so that X_2 is an open set in M. Any open set in a metric space is a countable union of closed sets, which implies that X_2 is a countable union of closed sets of topological dimension 0. Hence M is a countable union of closed sets of topological dimension 0, and therefore has topological dimension 0 as well. Alternatively, we can start with a pair of disjoint closed sets A, B in M, and apply the earlier argument to get disjoint closed sets A_1, B_1 in M satisfying (31.2). By normality, there are open subsets U_1, V_1 of M that satisfy (31.4), so that $\overline{U_1} \cap X_2$, $\overline{V_1} \cap X_2$ are disjoint relatively closed sets in X_2. The theorem in Section 29 implies that there are disjoint open sets W, Z in X_2 such that

$$\overline{U_1} \cap X_2 \subseteq W, \ \overline{V_1} \cap X_2 \subseteq Z, \text{ and } W \cup Z = X_2. \tag{31.8}$$

Thus $U_1 \cup W$, $V_1 \cup Z$ are disjoint open subsets of M such that

$$A \subseteq U_1 \cup W, \ B \subseteq V_1 \cup Z, \text{ and } (U_1 \cup W) \cup (V_1 \cup Z) = M. \tag{31.9}$$

32 Extrinsic conditions

Let M be a separable metric space, suppose that $X \subseteq M$ has topological dimension 0, and let A, B be disjoint closed subsets of M. We would like to show that there is an open set $W \subseteq M$ such that

$$A \subseteq W, \ \overline{W} \cap B = \emptyset, \text{ and } \partial W \cap X = \emptyset. \tag{32.1}$$

If X is a closed set in M, then one can get disjoint closed subsets A_1, B_1 of M as in (31.2), and use normality to get an open set $W \subseteq M$ such that

$$A_1 \subseteq W \text{ and } \overline{W} \cap B_1 = \emptyset, \tag{32.2}$$

and which therefore satisfies (32.1). If $H^1(X) = 0$, then one can get an open set W that satisfies (32.1) using the function ϕ in Section 26, as in Section 27.

By normality, there are open subsets U, V of M such that

$$A \subseteq U, \ B \subseteq V, \text{ and } \overline{U} \cap \overline{V} = \emptyset. \tag{32.3}$$

Thus $\overline{U} \cap X$, $\overline{V} \cap X$ are disjoint relatively closed subsets of X. The theorem in Section 29 implies that there are disjoint relatively open and closed subsets C, D of X such that

$$\overline{U} \cap X \subseteq C, \ \overline{V} \cap X \subseteq D, \text{ and } C \cup D = X. \tag{32.4}$$

In particular,

$$\overline{C} \cap D = C \cap \overline{D} = \emptyset \tag{32.5}$$

and

$$\overline{U} \cap D = C \cap \overline{V} = \emptyset. \tag{32.6}$$

The latter implies that $D \subseteq M \backslash U$, $C \subseteq M \backslash V$, and hence

$$\overline{D} \subseteq M \backslash U,\ \overline{C} \subseteq M \backslash V, \tag{32.7}$$

since U, V are open subsets of M. Equivalently,

$$U \cap \overline{D} = \overline{C} \cap V = \emptyset, \tag{32.8}$$

which implies that

$$A \cap \overline{D} = \overline{C} \cap B = \emptyset. \tag{32.9}$$

Note that

$$(A \cup C) \cap (B \cup D) = \emptyset. \tag{32.10}$$

Using (32.5) and (32.9), one can check that $A \cup C$ and $B \cup D$ are separated, i.e.,

$$(\overline{A \cup C}) \cap (B \cup D) = \emptyset,\ (A \cup C) \cap (\overline{B \cup D}) = \emptyset. \tag{32.11}$$

The strong version of normality for metric spaces implies that there is an open set $W \subseteq M$ such that

$$A \cup C \subseteq W,\ \overline{W} \cap (B \cup D) = \emptyset. \tag{32.12}$$

This implies (32.1), because $C \cup D = X$.

33 Level sets

Let $(M, d(x, y))$ be a metric space, and let f be a real-valued k-Lipschitz function on M. Suppose that $A \subseteq M$, $\alpha \geq 1$, and $H^{\alpha}_{con}(A) < +\infty$. Let $\{E_i\}_i$ be a collection of finitely or countably many subsets of M such that

$$A \subseteq \bigcup_i E_i \tag{33.1}$$

and

$$\sum_i (\operatorname{diam} E_i)^{\alpha} < +\infty. \tag{33.2}$$

For each i, let $\chi_i(r)$ be the characteristic function of $\overline{f(E_i)}$ on the real line, equal to 1 when $r \in \overline{f(E_i)}$ and 0 when $r \in \mathbf{R} \backslash \overline{f(E_i)}$. Consider

$$h(r) = \sum_i (\operatorname{diam} E_i)^{\alpha - 1}\, \chi_i(r). \tag{33.3}$$

This is a measurable function on $\mathbf{R}$, since each χ_i is measurable. If $|B|$ denotes the Lebesgue measure of $B \subseteq \mathbf{R}$, then

$$|\overline{f(E_i)}| \leq \operatorname{diam} \overline{f(E_i)} \leq k\ \operatorname{diam} E_i \tag{33.4}$$

for each i, and hence

$$\int_{\mathbf{R}} h(r)\,dr = \sum_i (\operatorname{diam} E_i)^{\alpha-1}\,|\overline{f(E_i)}| \le k \sum_i (\operatorname{diam} E_i)^{\alpha}. \tag{33.5}$$

For each $r \in \mathbf{R}$,

$$A \cap f^{-1}(r) \subseteq \bigcup \{E_i : \chi_i(r) = 1\}, \tag{33.6}$$

which implies that

$$H^{\alpha-1}_{con}(A \cap f^{-1}(r)) \le \sum \{(\operatorname{diam} E_i)^{\alpha-1} : \chi_i(r) = 1\} = h(r). \tag{33.7}$$

Moreover, if $\delta > 0$ and $\operatorname{diam} E_i < \delta$ for every i, then

$$H^{\alpha-1}_{\delta}(A \cap f^{-1}(r)) \le h(r) \tag{33.8}$$

for every $r \in \mathbf{R}$.

34 Level sets, 2

Let us continue with the same notations as in the previous section, and suppose also now that

$$H^{\alpha}(A) < \infty. \tag{34.1}$$

For each $j \ge 1$, let $\{E_{i,j}\}_i$ be a covering of A by finitely or countably many subsets of M such that

$$\operatorname{diam} E_{i,j} < 1/j \tag{34.2}$$

for every i, and

$$\sum_i (\operatorname{diam} E_{i,j})^{\alpha} < H^{\alpha}_{1/j}(A) + \frac{1}{j}. \tag{34.3}$$

Let $h_j(r)$ be the function on $\mathbf{R}$ corresponding to this covering as in the previous section. Thus

$$\int_{\mathbf{R}} h_j(r)\,dr \le k\Big(H^{\alpha}_{1/j}(A) + \frac{1}{j}\Big) \tag{34.4}$$

and

$$H^{\alpha-1}_{1/j}(A \cap f^{-1}(r)) \le h_j(r) \tag{34.5}$$

for every $r \in \mathbf{R}$ and $j \ge 1$. The latter implies that

$$H^{\alpha-1}(A \cap f^{-1}(r)) \le \liminf_{j \to \infty} h_j(r) \tag{34.6}$$

for every $r \in \mathbf{R}$. By Fatou's lemma,

$$\int_{\mathbf{R}} \liminf_{j \to \infty} h_j(r)\,dr \le k\,H^{\alpha}(A). \tag{34.7}$$

In particular, if $H^{\alpha}(A) = 0$, then it follows that

$$H^{\alpha-1}(A \cap f^{-1}(r)) = 0 \tag{34.8}$$

for almost every $r \in \mathbf{R}$.

35 Topological dimension $\leq n$

A separable metric space M is said to have *topological dimension* $\leq n$ if for every $p \in M$ and $r > 0$ there is an open set U in M such that $p \in U \subseteq B(p,r)$ and ∂U has topological dimension $\leq n-1$. Thus the definition proceeds inductively, and one can include the $n = 0$ case by considering the empty set to have topological dimension -1. More precisely, this definition also applies to any subset X of M, using the restriction of the metric on M to X. If M has topological dimension $\leq n$, then one can check that every $X \subseteq M$ has topological dimension $\leq n$, using induction.

If $H^{n+1}(M) = 0$ and $p \in M$, then the discussion in the previous section applied to $\alpha = n+1$ and $f_p(x) = d(p,x)$ implies that

$$H^n(\{x \in M : d(p,x) = r\}) = 0 \tag{35.1}$$

for almost every $r > 0$. This implies that M has topological dimension $\leq n$, because $\partial B(p,r)$ has topological dimension $\leq n-1$ for every $p \in M$ and almost every $r > 0$, by induction. In particular, the topological dimension of M is less than or equal to the Hausdorff dimension of M, since $H^\alpha(M) = 0$ when $\alpha > \dim_H M$.

A well-known theorem states that a separable metric space has topological dimension $\leq n$ when it is the union of a sequence of closed subsets with topological dimension $\leq n$, extending the $n = 0$ case discussed in Section 31. As a consequence, a separable metric space M has topological dimension $\leq n$ when $M = X \cup Y$, X and Y have topological dimension $\leq n$, and X is closed. For $M \backslash X$ is an open set with topological dimension $\leq n$ in this case, and one can argue as in Section 31 that M is a countable union of closed sets with topological dimension $\leq n$.

If X, Y are arbitrary subsets of a separable metric space with topological dimensions $\leq m, n$, respectively, then it can be shown that

(35.2) the topological dimension of $X \cup Y$ is $\leq m + n + 1$.

This estimate is sharp, as in the example of the sets of rational and irrational numbers.

Another well-known theorem asserts that a separable metric space M with topological dimension $\leq n$ can be expressed as the union of two subsets X and Y, where X has topological dimension $\leq n-1$, and Y has topological dimension 0. Under these conditions, there is a countable base for the topology of M consisting of open sets whose boundaries have topological dimension $\leq n-1$. If X is the union of the boundaries of the open sets in this countable base, then X has topological dimension $\leq n-1$, since it is the countable union of closed sets with topological dimension $\leq n-1$. It is easy to see that $Y = M \backslash X$ has topological dimension 0, by construction. Note that the theorem about countable unions of closed sets is applied to sets of topological dimension $\leq n-1$ in this argument, which permits the conclusion to be used in the analysis of countable unions of closed sets of topological dimension $\leq n$, by induction.

By repeating the process, M can be realized as the union of $n+1$ subsets of topological dimension 0. This is consistent with the estimate (35.2) for the topological dimension of the union of arbitrary subsets of M. The latter can also be used to show that M is the union of a pair of sets of topological dimension $\le l, r$, respectively, when $n = l + r + 1$.

If $H^{n+1}(M) = 0$, then there is a set $X \subseteq M$ such that $H^n(X) = 0$ and $M \backslash X$ has topological dimension 0. This simply uses countable subadditivity of Hausdorff measure instead of the theorem about topological dimensions of countable unions of closed sets.

36 Extrinsic conditions, 2

Let M be a metric space, let X be a subset of M, and let E be a relatively open subset of X. Of course, $X \backslash \overline{E}$ is a relatively open set in X that is disjoint from E. This implies that E and $X \backslash \overline{E}$ are separated as subsets of X and hence in M, in the sense that neither contains a limit point of the other. By the strong version of normality for metric spaces, there are disjoint open sets U, V in M such that

$$E \subseteq U, \ X \backslash \overline{E} \subseteq V. \tag{36.1}$$

Let $\partial_X E$ be the boudary of E relative to X, consisting of limit points of E in X that are not contained in E. It is easy to see that

$$\partial U \cap X \subseteq \partial_X E, \tag{36.2}$$

since U is contained in the closed set $M \backslash V$ and hence $\overline{U} \subseteq M \backslash V$. If W is another open set in M such that $E \subseteq W \subseteq U$, then we also have that

$$\partial W \cap X \subseteq \partial_X E. \tag{36.3}$$

Suppose now that M is separable and X has topological dimension $\le n$. Using the remarks in the preceding paragraph, one can check that each $p \in X$ has arbitrarily small neighborhoods W in M such that $\partial W \cap X$ has topological dimension $\le n-1$. This actually works for every $p \in M$, because $X \cup \{p\}$ also has topological dimension $\le n$. For every $p \in M$ and $r > 0$,

$$\partial B(p, r) \subseteq \{x \in M : d(p, x) = r\}. \tag{36.4}$$

If $H^{n+1}(X) = 0$, then $H^n(\partial B(p, r) \cap X) = 0$ for almost every $r > 0$, and one can take $W = B(p, r)$ for such r in this case.

Note that a separable metric space M has topological dimension $\le n$ if and only if for every $p \in M$ and closed set $B \subseteq M$ with $p \notin B$ there is an open set U in M such that $p \in U$, $\overline{U} \cap B = \emptyset$, and ∂U has topological dimension $\le n-1$. A well-known theorem states that if M has topological dimension $\le n$ and A, B are disjoint closed subsets of M, then there is an open set U in M such that $A \subseteq U$, $\overline{U} \cap B = \emptyset$, and ∂U has topological dimension $\le n-1$. The extrinsic version of this theorem asserts that if $X \subseteq M$ has topological dimension $\le n$

and A, B are disjoint closed subsets of M, then there is an open set $U \subseteq M$ such that $A \subseteq U$, $\overline{U} \cap B = \emptyset$, and $\partial U \cap X$ has topological dimension $\leq n-1$. If $X \subseteq M$ satisfies $H^{n+1}(X) = 0$ and A, B are disjoint nonempty closed subsets of M, then one can use the function ϕ in Section 26 to show that there is an open set $U \subseteq M$ such that $A \subseteq U$, $\overline{U} \cap B = \emptyset$, and $H^n(\partial U \cap X) = 0$.

37 Intersections

Let M be a separable metric space with topological dimension $\leq n-1$, and let $A_1, B_1, \ldots, A_n, B_n$ be n disjoint pairs of closed subsets of M. By repeating the extrinsic separation theorem mentioned in the previous section, it is easy to see that there are n open subsets $U_1, \ldots, U_n$ of M such that $A_i \subseteq U_i$ and $\overline{U_i} \cap B_i = \emptyset$ for each i, and

(37.1) $$\partial U_1 \cap \partial U_2 \cap \cdots \cap \partial U_n = \emptyset.$$

Using the Brouwer fixed-point theorem, one can then show that $\mathbf{R}^n$ does not have topological dimension $\leq n-1$. Of course, $\mathbf{R}^n$ does have topological dimension $\leq n$.

Here is a slightly different way to look at the case where $H^n(M) = 0$. Let ϕ_i be the function associated to the pair A_i, B_i as in Section 26 for $i = 1, \ldots, n$, and let

(37.2) $$\Phi = (\phi_1, \ldots, \phi_n)$$

be the combined mapping from M into $\mathbf{R}^n$. If $\Lambda^i_\epsilon \subseteq M$ is associated to A_i, B_i as before, then the restriction of Φ to

(37.3) $$\Lambda^1_\epsilon \cap \cdots \cap \Lambda^n_\epsilon$$

is Lipschitz for each $\epsilon > 0$. Of course,

(37.4) $$H^n(\Lambda^1_\epsilon \cap \cdots \cap \Lambda^n_\epsilon) \leq H^n(M) = 0$$

for each $\epsilon > 0$, and hence

(37.5) $$H^n(\Phi(\Lambda^1_\epsilon \cap \cdots \cap \Lambda^n_\epsilon)) = 0.$$

This implies that

(37.6) $$H^n(\Phi(M)) \leq \sum_{j=1}^{\infty} H^n(\Phi(\Lambda^1_{1/j} \cap \cdots \cap \Lambda^n_{1/j})) = 0.$$

Since the open unit cube $(0,1)^n$ has positive n-dimensional Hausdorff measure, we get that

(37.7) $$(0,1)^n \not\subseteq \Phi(M).$$

If $r = (r_1, \ldots, r_n) \in (0,1)^n \backslash \Phi(M)$, then

(37.8) $$U_i = \{x \in M : \phi_i(x) < r_i\}$$

has the properties mentioned in the previous paragraph.

38 Embeddings

Let M be a separable metric space with topological dimension $\le n$. A famous theorem states that M is homeomorphic to a bounded set in $\mathbf{R}^{2n+1}$. Moreover, this set in $\mathbf{R}^{2n+1}$ may be taken to have the property that its closure has Hausdorff dimension $\le n$. In particular, there is a metric on M that determines the same topology and with respect to which M has Hausdorff dimension $\le n$. Of course, the proof of this theorem relies heavily on the theory of the topological dimension, and so cannot be used to derive basic results about the topological dimension from properties of Hausdorff measure. In some situations, changing the metric on one part may not say much about the rest. For example, this theorem implies that there is a topologically-equivalent metric on the set of irrational numbers with Hausdorff dimension 0, and 1-dimensional Hausdorff measure 0 in particular. The set of rational numbers already has these properties with respect to the standard metric, but the real line still has topological dimension 1.

If M is already embedded in some $\mathbf{R}^l$, then it may not be possible to deform this embedding using global homeomorphisms on $\mathbf{R}^l$ to one with Hausdorff dimension equal to the topological dimension as in the previous paragraph, even when M is compact. See [16, 66, 83] for more information.

39 Local Lipschitz conditions

Let $(M_1, d_1(x, y))$ and $(M_2, d_2(u, v))$ be metric spaces. Let us say that a mapping $f : M_1 \to M_2$ is *locally k-Lipschitz at scale δ* for some $k \ge 0$, $\delta > 0$ if

$$d_2(f(x), f(y)) \le k\, d_1(x, y) \tag{39.1}$$

for every $x, y \in M_1$ such that

$$d_1(x, y) < \delta. \tag{39.2}$$

As in Section 17, if f is locally k-Lipschitz at the scale of δ, $A \subseteq M_1$, and $\alpha \ge 0$, then

$$H^\alpha_{k\delta'}(f(A)) \le k^\alpha\, H^\alpha_{\delta'}(A) \tag{39.3}$$

when $0 < \delta' < \delta$, and hence

$$H^\alpha(f(A)) \le k^\alpha\, H^\alpha(A). \tag{39.4}$$

There are versions of the statements in Sections 33 and 34 for local Lipschitz conditions as well.

For example, suppose that M is a compact connected C^1 submanifold of $\mathbf{R}^n$. Let $d_1(x, y)$ be the ordinary Euclidean distance restricted to $x, y \in M$, and let $d_2(x, y)$ be the Riemannian distance on M, which is to say the length of the shortest path on M connecting x and y. The latter is always greater than or equal to the former, so that the identity mapping on M is 1-Lipschitz as a mapping from $(M, d_2(x, y))$ onto $(M, d_1(x, y))$. The identity mapping on M is also Lipschitz as a mapping from $(M, d_1(x, y))$ onto $(M, d_2(x, y))$, and more precisely it is locally Lipschitz at the scale of δ with a constant $k(\delta) \to 1$ as $\delta \to 0$. It follows that Hausdorff measures on M with respect to these two metrics are the same.

A mapping between metric spaces may be described as *locally Lipschitz* if each point in the domain has a neighborhood on which the mapping is Lipschitz. This permits both the size of the neighborhood and the Lipschitz constant to depend on the point. It is easy to see that the restriction of a locally Lipschitz mapping to a compact set is Lipschitz. Similarly, a mapping is *locally k-Lipschitz* for some $k \geq 0$ if each element of the domain has a neighborhood on which the mapping is k-Lipschitz. If f is locally k-Lipschitz and $A \subseteq M_1$ is compact, then f is locally k-Lipschitz at the scale of δ on A for some $\delta > 0$. More precisely, one can cover A by open balls $B(p, r)$ such that the restriction of f to $B(p, 2\,r)$ is k-Lipschitz. Compactness implies that A can be covered by finitely many such balls $B(p_1, r_1), \ldots, B(p_n, r_n)$, and one can take $\delta = \min(r_1, \ldots, r_n)$.

40 Other Lipschitz conditions

Let $(M_1, d_1(x, y))$, $(M_2, d_2(u, v))$ be metric spaces again, and let a be a positive real number. A mapping $f : M_1 \to M_2$ is said to be *Lipschitz of order a* if there is a $k > 0$ such that

$$d_2(f(x), f(y)) \leq k\, d_1(x, y)^a \tag{40.1}$$

for every $x, y \in M_1$. Of course, this reduces to the ordinary Lipschitz condition when $a = 1$. This condition is also known as *Hölder continuity of order a*, and one can consider local versions as well, as in the previous section. In some situations, such as for real-valued functions on Euclidean spaces, quite different conditions are used for $a > 1$, related to the regularity of the derivatives of f.

If $a \leq 1$, then f is Lipschitz of order a as a mapping

$$(M_1, d_1(x, y)) \to (M_2, d_2(u, v)) \tag{40.2}$$

if and only if it is Lipschitz of order 1 as a mapping

$$(M_1, d_1(x, y)^a) \to (M_2, d_2(u, v)). \tag{40.3}$$

This works for any $a > 0$ when $d_1(x, y)$ is an ultrametric on M_1, but otherwise $d_1(x, y)^a$ may not be a metric on M_1. If $d_1(x, y)$ is a snowflake metric, so that

$$d_1(x, y) = d(x, y)^t \tag{40.4}$$

for some metric $d(x, y)$ on M and $t \in (0, 1)$, then this works for $a \leq 1/t$. Alternatively, f is Lipschitz of order $a \geq 1$ with respect to the initial metrics as in (40.2) if and only if it is Lipschitz of order 1 as a mapping

$$(M_1, d_1(x, y)) \to (M_2, d_2(u, v)^{1/a}). \tag{40.5}$$

The same statement holds for every $a > 0$ when $d_2(u, v)$ is an ultrametric, and for the appropriate range of a when $d_2(u, v)$ is a snowflake metric.

In particular, the estimates for Hausdorff measures in Section 17 carry over to Lipschitz mappings of any order. Specifically, if f is k-Lipschitz of order a and $E \subseteq M_1$ is bounded, then $f(E)$ is bounded in M_2, and

$$\operatorname{diam} f(E) \le k\, (\operatorname{diam} E)^a. \tag{40.6}$$

This implies that

$$H^\alpha(f(A)) \le k^\alpha\, H^{a\alpha}(A) \tag{40.7}$$

for every $A \subseteq M_1$ and $\alpha \ge 0$, and hence

$$\dim_H f(A) \le \frac{\dim_H A}{a}. \tag{40.8}$$

41 Quasimetrics

A *quasimetric* on a set M is a symmetric nonnegative real-valued function $d(x, y)$ defined for $x, y \in M$ such that $d(x, y) = 0$ if and only if $x = y$ and

$$d(x, z) \le C\, (d(x, y) + d(y, z)) \tag{41.1}$$

for some $C \ge 1$ and all $x, y, z \in M$. Thus $d(x, y)$ is a metric on M when $C = 1$. If $d(x, y)$ is a quasimetric on M, then

$$d(x, y)^t \tag{41.2}$$

is also a quasimetric for every $t > 0$. In particular, $d(x, y)^t$ is a quasimetric when $d(x, y)$ is a metric and $t > 1$. Lipschitz conditions can be extended to mappings between sets equipped with quasimetrics instead of metrics in the obvious way. The order of a Lipschitz condition can be changed by changing the quasimetrics on the domain or the range, is in the previous section. This is a bit simpler for quasimetrics, since arbitrary $t > 0$ are allowed in (41.2). However, an advantage of ordinary metrics is that they determine real-valued Lipschitz functions of order 1. If $d(x, y)$ is a quasimetric on M, then it is shown in [48] that there is a metric $\rho(x, y)$ on M and $C_1, \delta > 0$ such that

$$C_1^{-1}\, \rho(x, y) \le d(x, y)^\delta \le C_1\, \rho(x, y) \tag{41.3}$$

for every $x, y \in M$.

42 Locally flat mappings

Let $(M_1, d_1(x, y))$, $(M_2, d_2(u, v))$ be metric spaces again. Let us say that a mapping $f : M_1 \to M_2$ is *locally flat* if it is locally Lipschitz, and for each $p \in M_1$ the Lipschitz constant of the restriction of f to $B(p, r)$ converges to 0 as $r \to 0$. Equivalently, f is locally flat if for each $p \in M_1$ and $\epsilon > 0$ there is an $r > 0$ such that the restriction of f to $B(p, r)$ is ϵ-Lipschitz, which is the same as saying that f is locally ϵ-Lipschitz for each $\epsilon > 0$. Similarly, let us say that f is *uniformly locally flat* if it is locally $k(\delta)$-Lipschitz at the scale of δ for sufficiently small $\delta > 0$, where

$$\lim_{\delta \to 0} k(\delta) = 0. \tag{42.1}$$

Thus uniformly locally flat mappings are locally flat, and the restriction of a locally flat mapping to a compact set is uniformly locally flat.

A real-valued locally flat function on $\mathbf{R}^n$ has differential equal to 0 at each point. If $f : M_1 \to M_2$ is k-Lipschitz of order $a > 1$, then f is uniformly locally flat, with $k(\delta) = k\, \delta^{a-1}$. Of course, locally constant mappings are locally flat, and may not be constant on disconnected spaces. Snowflake spaces can have nonconstant Lipschitz functions of order $a > 1$ even when they are connected.

If f is uniformly locally flat, $A \subseteq M$, and $H^\alpha(A) < +\infty$ for some $\alpha \geq 0$, then

$$H^\alpha(f(A)) = 0. \tag{42.2}$$

The same statement holds when f is locally flat and A is also compact. If A is connected, then $f(A)$ is connected, and

$$\operatorname{diam} f(A) \leq H^1(f(A)), \tag{42.3}$$

as in Section 18. Hence f is constant on A when A is connected and $H^1(A)$ is finite. The same argument works for Lipschitz mappings of order $a > 1$ when A is connected and $H^a(A) = 0$.

43 Rectifiable curves

Let a, b be real numbers with $a \leq b$, let $(M, d(u, v))$ be a metric space, and let $p(t)$ be a continuous path in M defined on the closed interval $[a, b]$. If $\mathcal{P} = \{t_j\}_{j=0}^n$ is a partition of $[a, b]$, so that

$$a = t_0 < t_1 < \cdots < t_n = b, \tag{43.1}$$

then put

$$\Lambda(\mathcal{P}) = \sum_{j=1}^{n} d(p(t_j), p(t_{j-1})). \tag{43.2}$$

The path $p(t)$, $a \leq t \leq b$, is said to have *finite length* if the $\Lambda(\mathcal{P})$'s are bounded, in which case the length of the path is defined by

$$\Lambda_a^b = \sup_{\mathcal{P}} \Lambda(\mathcal{P}), \tag{43.3}$$

where the supremum is taken over all partitions $\mathcal{P}$ of $[a, b]$.

For example, if $p : [a, b] \to M$ is k-Lipschitz, then

$$\Lambda(\mathcal{P}) \leq k\,(b - a) \tag{43.4}$$

for every partition $\mathcal{P}$ of $[a,b]$, and hence p has finite length

$$\Lambda_a^b \le k\,(b-a). \tag{43.5}$$

In particular, constant paths have length 0. Conversely, for any path p,

$$d(p(x),p(y)) \le \Lambda(\mathcal{P}) \tag{43.6}$$

when $a \le x \le y \le b$ and $\mathcal{P}$ is the partition consisting of these four points, which implies that

$$\operatorname{diam} p([a,b]) \le \Lambda_a^b. \tag{43.7}$$

This shows that paths with length 0 are constant.

If $a \le x \le y \le b$, then every partition of $[x,y]$ can be extended to a partition of $[a,b]$. Hence the restriction of a continuous path $p : [a,b] \to M$ of finite length to $[x,y]$ also has finite length, and

$$\Lambda_x^y \le \Lambda_a^b. \tag{43.8}$$

A partition $\mathcal{P} = \{t_j\}_{j=0}^n$ of $[a,b]$ is said to be a *refinement* of another partition $\mathcal{P}' = \{r_i\}_{i=0}^l$ if for each $i = 0,1,\ldots,l$ there is a $j = 0,1,\ldots,n$ such that $r_i = t_j$. Using the triangle inequality, one can check that

$$\Lambda(\mathcal{P}') \le \Lambda(\mathcal{P}) \tag{43.9}$$

when $\mathcal{P}$ is a refinement of $\mathcal{P}'$. If p has finite length and $a \le x \le b$, then it follows that

$$\Lambda_a^b = \Lambda_a^x + \Lambda_x^b. \tag{43.10}$$

Indeed, $\Lambda_a^x + \Lambda_x^b \le \Lambda_a^b$ because arbitrary partitions of $[a,x]$ and $[x,b]$ can be combined to get a partition of $[a,b]$. The opposite inequality holds because any partition of $[a,b]$ can be refined to include x, and the refinement is then a combination of partitions of $[a,x]$ and $[x,b]$. Similar reasoning shows that $p : [a,b] \to M$ has finite length when the restrictions of p to $[a,x]$ and $[x,b]$ have finite length.

In totally disconnected spaces, continuous paths are automatically constant. Snowflake spaces may contain many nontrivial continuous paths, but one can check that continuous paths of finite length are constant. However, there are fractal sets such as Sierpinski gaskets and carpets and Menger sponges with numerous nontrivial continuous paths of finite length. Smooth manifolds also have plenty of continuous paths of finite length, and this can be extended to sub-Riemannian spaces as well.

44 Length and measure

Let $(M, d(u,v))$ be a metric space, and let $p : [a,b] \to M$ be a continuous path of finite length. If $\mathcal{P} = \{t_j\}_{j=0}^n$ is any partition of $[a,b]$, then

$$\sum_{j=1}^n \operatorname{diam} p([t_{j-1},t_j]) \le \sum_{j=1}^n \Lambda_{t_{j-1}}^{t_j} = \Lambda_a^b, \tag{44.1}$$

by the computations in the previous section. If in addition $\delta > 0$ and

$$\operatorname{diam} p([t_{j-1}, t_j]) < \delta, \ 1 \le j \le n, \tag{44.2}$$

then it follows that

$$H^1_\delta(p([a, b])) \le \Lambda_a^b. \tag{44.3}$$

Because of uniform continuity, (44.2) holds for any $\delta > 0$ when $\mathcal{P}$ is a sufficiently fine partition of $[a, b]$, which implies that

$$H^1(p([a, b])) \le \Lambda_a^b. \tag{44.4}$$

If $p : [a, b] \to M$ is also injective, then

$$H^1(p([a, b])) = \Lambda_a^b. \tag{44.5}$$

To see this, observe that

$$d(p(x), p(y)) \le \operatorname{diam} p([x, y]) \le H^1(p([x, y])) \tag{44.6}$$

when $a \le x \le y \le b$, since $p([x, y])$ is connected. If $I_1, \ldots, I_l$ are pairwise disjoint closed subintervals of $[a, b]$, then $p(I_1), \ldots, p(I_l)$ are pairwise disjoint compact subsets of M, and are therefore at positive distance from each other. Thus

$$\sum_{i=1}^{l} H^1(p(I_i)) = H^1\Big(\bigcup_{i=1}^{l} p(I_i)\Big) \le H^1(p([a, b])), \tag{44.7}$$

using the additivity of Hausdorff measure in this case, as in Section 11. If

$$a \le x_1 < y_1 < x_2 < y_2 < x_3 < \cdots < y_l \le b, \tag{44.8}$$

then we can apply these estimates to $I_i = [x_i, y_i]$ to get that

$$\sum_{i=1}^{l} d(p(x_i), p(y_i)) \le H^1(p([a, b])). \tag{44.9}$$

This also holds when

$$a \le x_1 < y_1 \le x_2 < y_2 \le x_3 < \cdots < y_l \le b, \tag{44.10}$$

by passing to suitable limits. Hence

$$\Lambda(\mathcal{P}) \le H^1(p([a, b])) \tag{44.11}$$

for every partition $\mathcal{P}$ of $[a, b]$, as desired. Note that an injective continuous path $p : [a, b] \to M$ has finite length when $H^1(p([a, b])) < +\infty$, by the same argument.

This argument can also be extended to deal with paths with only finitely many crossings, for instance. However, strict inequality can occur in (44.4) for arbitrary continuous paths of finite length, as when such a path retraces an arc.

45 Mappings of paths

Let $(M_1, d_1(u,v))$ and $(M_2, d_2(w,z))$ be metric spaces, and let $p : [a,b] \to M_1$ and $f : M_1 \to M_2$ be continuous mappings. Thus $\widetilde{p} = f \circ p : [a,b] \to M_2$ is also continuous. For each partition $\mathcal{P}$ of $[a,b]$, let $\Lambda(\mathcal{P})$, $\widetilde{\Lambda}(\mathcal{P})$ be the corresponding sums for the paths p, $\widetilde{p}$, respectively, as in Section 43.

If $f : M_1 \to M_2$ is k-Lipschitz for some $k \geq 0$, then

$$\widetilde{\Lambda}(\mathcal{P}) \leq k\,\Lambda(\mathcal{P}) \tag{45.1}$$

for every partition $\mathcal{P}$ of $[a,b]$. Hence $\widetilde{p}$ has finite length when p does, and

$$\widetilde{\Lambda}_a^b \leq k\,\Lambda_a^b, \tag{45.2}$$

where Λ_a^b, $\widetilde{\Lambda}_a^b$ are the lengths of p, $\widetilde{p}$, respectively.

If $f : M_1 \to M_2$ is locally k-Lipschitz at the scale of δ for some $\delta > 0$, then we get the same estimate for $\widetilde{\Lambda}(\mathcal{P})$ when the partition $\mathcal{P} = \{t_j\}_{j=0}^n$ is sufficiently fine so that

$$d_1(p(t_j), p(t_{j-1})) < \delta,\ j = 1, \ldots, n. \tag{45.3}$$

Every partition of $[a,b]$ has a refinement with this property, because of uniform continuity. It follows again that $\widetilde{p}$ has finite length when p does, and with the same estimate for the length.

If $f : M_1 \to M_2$ is locally k-Lipschitz, then the restriction of f to $p([a,b])$ is locally k-Lipschitz at the scale of δ for some $\delta > 0$, since $p([a,b])$ is compact. Hence the same conclusions hold in this case. If $f : M_1 \to M_2$ is locally flat, then $\widetilde{p}$ has length 0, and is therefore constant.

46 Reparameterizations

Let $(M, d(u,v))$ be a metric space, and let $p : [a,b] \to M$ be a continuous mapping. Also let ϕ be a strictly increasing continuous mapping from another closed interval $[\widehat{a}, \widehat{b}]$ onto $[a,b]$, so that $\widehat{p} = p \circ \phi : [\widehat{a}, \widehat{b}] \to M$ is a continuous path in M. If $\widehat{P} = \{\widehat{t}_j\}_{j=0}^n$ is a partition of $[\widehat{a}, \widehat{b}]$, then $\mathcal{P} = \{\phi(\widehat{t}_j)\}_{j=0}^n$ is a partition of $[a,b]$, and every partition of $[a,b]$ corresponds to a partition $[\widehat{a}, \widehat{b}]$ in this way. By construction,

$$\sum_{j=1}^n d(\widehat{p}(\widehat{t}_j), \widehat{p}(\widehat{t}_{j-1})) = \sum_{j=1}^n d(p(\phi(\widehat{t}_j)), p(\phi(\widehat{t}_{j-1}))), \tag{46.1}$$

which implies that $\widehat{p}$ has finite length if and only if p has finite length, and that their lengths are the same in this case.

The same conclusion holds when ϕ is a monotone increasing continuous mapping from $[\widehat{a}, \widehat{b}]$ onto $[a,b]$. The correspondence between partitions of $[\widehat{a}, \widehat{b}]$ and $[a,b]$ is a bit more complicated when ϕ is not strictly increasing, but this is not significant for the approximations of the lengths of $\widehat{p}$ and p. This is because ϕ and hence $\widehat{p}$ is constant on any interval $[x,y] \subseteq [\widehat{a}, \widehat{b}]$ such that $\phi(x) = \phi(y)$.

If $p : [a, b] \to M$ has finite length, then the length Λ_a^r of the restriction of p to $[a, r]$ is a monotone increasing function of r on $[a, b]$. One can also show that Λ_a^r is continuous in r, as follows. Let $\epsilon > 0$ be given, and let $\mathcal{P}$ be a partition of $[a, b]$ such that

$$\Lambda_a^b < \Lambda(\mathcal{P}) + \epsilon. \tag{46.2}$$

Suppose that $a \le r < t \le b$ and that $\mathcal{P}$ does not contain any element of (r, t). Let $\mathcal{P}(r, t)$ be a refinement of $\mathcal{P}$ that contains r, t as consecutive terms. Thus

$$\Lambda(P) \le \Lambda(\mathcal{P}(r, t)) \le \Lambda_a^r + d(p(r), p(t)) + \Lambda_t^b, \tag{46.3}$$

since the rest of $\mathcal{P}(r, t)$ partitions $[a, r]$ and $[t, b]$. This implies that

$$\Lambda_r^t \le d(p(r), p(t)) + \epsilon, \tag{46.4}$$

because $\Lambda_a^b = \Lambda_a^r + \Lambda_r^t + \Lambda_t^b$. This permits Λ_r^t to be estimated in terms of the continuity of p when r, t are sufficiently close.

If $a \le r < t \le b$ and $\Lambda_a^r = \Lambda_a^t$, then $\Lambda_r^t = 0$, and p is constant on $[r, t]$. It follows that there is a mapping $q : [0, \Lambda_a^b] \to M$ such that $p(r) = q(\Lambda_a^r)$ for each $r \in [a, b]$. Moreover, q is 1-Lipschitz, because $d(p(r), p(t)) \le \Lambda_r^t$ when $a \le r < t \le b$. The length of q is equal to the length Λ_a^b of p, by the earlier remarks about arbitrary reparameterizations.

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