Some topics related to metrics and norms, including ultrametrics and ultranorms, 4

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Abstract

Some topics related to Lipschitz mappings are discussed, where the corresponding metrics may change in a certain way. This includes bounded linear mappings between vector spaces over fields with absolute value functions, where the corresponding norms may change in a similar way.

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Part I
Basic notions

1 q-Semimetrics

Let X be a set, and q be a positive real number. A nonnegative real-valued function $d(x, y)$ defined for $x, y \in X$ is said to be a *q-semimetric* on X if it satisfies the following three conditions. First,

$$d(x, x) = 0 \tag{1.1}$$

for every $x \in X$. Second, $d(x, y)$ should be symmetric in x and y, so that

$$d(x, y) = d(y, x) \tag{1.2}$$

for every $x, y \in X$. Third, we ask that

$$d(x, z)^q \le d(x, y)^q + d(y, z)^q \tag{1.3}$$

for every $x, y, z \in X$, which is the q-semimetric version of the triangle inequality. If we also have that

$$d(x, y) > 0 \tag{1.4}$$

for every $x, y \in X$ with $x \neq y$, then $d(x, y)$ is said to be a *q-metric* on X. If $q = 1$, then this reduces to the usual notion of a semimetric and a metric on X. Note that $d(x, y)$ is a q-semimetric or q-metric on X exactly when $d(x, y)^q$ is an ordinary semimetric or metric on X, respectively. Observe too that (1.3) can be reformulated equivalently as saying that

$$d(x, z) \le (d(x, y)^q + d(y, z)^q)^{1/q} \tag{1.5}$$

for every $x, y, z \in X$.

Clearly

$$\max(r, t) \le (r^q + t^q)^{1/q} \tag{1.6}$$

for every positive real number q and nonnegative real numbers r, t. If q_1, q_2 are positive real numbers with $q_1 \le q_2$, then we get that

$$\begin{aligned} r^{q_2} + t^{q_2} &\le \max(r, t)^{q_2 - q_1} (r^{q_1} + t^{q_1}) \\ &\le (r^{q_1} + t^{q_1})^{(q_2 - q_1)/q_1 + 1} = (r^{q_1} + t^{q_1})^{q_2/q_1} \end{aligned} \tag{1.7}$$

for every $r, t \ge 0$. This implies the well-known inequality

$$(r^{q_2} + t^{q_2})^{1/q_2} \le (r^{q_1} + t^{q_1})^{1/q_1} \tag{1.8}$$

for $r, t \geq 0$ when $0 < q_1 \leq q_2 < \infty$. If $d(x, y)$ is a q_2-semimetric on X for some $q_2 > 0$, then it follows that $d(x, y)$ is a q_1-semimetric on X as well when $0 < q_1 \leq q_2$, using the reformulation (1.5) of the q-semimetric version of the triangle inequality. Of course, there is an analogous statement for q-metrics.

A nonnegative real-valued function $d(x, y)$ defined for $x, y \in X$ is said to be a *semi-ultrametric* on X if it satisfies (1.1), (1.2), and

$$d(x, z) \leq \max(d(x, y), d(y, z)) \tag{1.9}$$

for every $x, y, z \in X$, instead of (1.3). If $d(x, y)$ satisfies (1.4) too, then $d(x, y)$ is said to be an *ultrametric* on X. One can check that a semi-ultrametric on X is a q-semimetric on X for every $q > 0$, and similarly for ultrametrics. Remember that the *discrete metric* on X is defined by putting

$$d(x, y) = 1 \tag{1.10}$$

for every $x, y \in X$ with $x \neq y$, and using (1.1) otherwise. It is easy to see that this defines an ultrametric on X.

Observe that

$$(r^q + t^q)^{1/q} \leq 2^{1/q} \max(r, t) \tag{1.11}$$

for every $q > 0$ and $r, t \geq 0$. Combining this with (1.6), we get that

$$\lim_{q \to \infty} (r^q + t^q)^{1/q} = \max(r, t) \tag{1.12}$$

for every $r, t \geq 0$. Thus semi-ultrametrics and ultrametrics may be considered as analogues of q-semimetrics and q-metrics with $q = \infty$. It will sometimes be convenient to refer to the range $0 < q \leq \infty$ in this way.

2 Open balls

Let X be a set, and let $d(x, y)$ be a q-semimetric on X for some $q > 0$. The *open ball* in X centered at a point $x \in X$ with radius $r > 0$ with respect to $d(\cdot, \cdot)$ can be defined as usual by

$$B(x, r) = B_d(x, r) = \{y \in X : d(x, y) < r\}. \tag{2.1}$$

If $z \in B(x, r)$, $t > 0$, and

$$d(x, z)^q + t^q \leq r^q, \tag{2.2}$$

then it is easy to see that

$$B(z, t) \subseteq B(x, r), \tag{2.3}$$

using the q-semimetric version of the triangle inequality (1.3). More precisely, if $z \in B(x, r)$, then $d(x, z)^q < r^q$, and so

$$t = (r^q - d(x, z)^q)^{1/q} \tag{2.4}$$

is a positive real number. In this case, equality holds in (2.2), and so (2.3) holds with this choice of t.

Note that

$$B_{d^q}(x, r^q) = B_d(x, r) \tag{2.5}$$

for each $x \in X$ and $r > 0$. Thus one could also reduce (2.3) to its analogue for $d(\cdot, \cdot)^q$, which is an ordinary semimetric on X, as in the previous section. If $d(\cdot, \cdot)$ is a semi-ultrametric on X, then we have that

$$B(z, r) \subseteq B(x, r) \tag{2.6}$$

when $d(x, z) < r$. This may be considered as the $q = \infty$ version of the remarks in the preceding paragraph, but it is easier to derive (2.6) directly from the ultrametric version (1.9) of the triangle inequality. It follows that

$$B(x, r) = B(z, r) \tag{2.7}$$

when $d(x, z) < r$, because (2.6) also holds with the roles of x and z exchanged.

Let $d(\cdot, \cdot)$ be a q-semimetric on X for any $q > 0$ again. As usual, a subset U of X is said to be an *open set* with respect to $d(\cdot, \cdot)$ if for each $x \in U$ there is an $r > 0$ such that

$$B(x, r) \subseteq U. \tag{2.8}$$

It is easy to see that this defines a topology on X, and the earlier discussion of (2.3) implies that open balls in X with respect to $d(\cdot, \cdot)$ are open sets. This topology on X associated to $d(\cdot, \cdot)$ is the same as the topology associated to $d(\cdot, \cdot)^q$, because of (2.5), which permits one to reduce to the case of ordinary semimetrics. One can check that this topology on X associated to $d(\cdot, \cdot)$ is Hausdorff exactly when $d(\cdot, \cdot)$ is a q-metric on X.

Suppose that $d(\cdot, \cdot)$ is a semi-ultrametric on X, and that

$$B(x, r) \cap B(z, r) \neq \emptyset \tag{2.9}$$

for some $x, z \in X$ and $r > 0$. This implies that

$$d(x, z) < r, \tag{2.10}$$

by the ultrametric version of the triangle inequality. It follows that (2.7) holds under these conditions, as before. In particular, if z is a limit point of $B(x, r)$ with respect to the topology determined by $d(\cdot, \cdot)$ on X, then z is an element of $B(x, r)$. Thus $B(x, r)$ is a closed set in X with respect to this topology in this situation.

3 Closed balls

Let $d(x, y)$ be a q-semimetric on a set X for some $q > 0$ again. If $x \in X$ and r is a nonnegative real number, then the *closed ball* in X centered at x with radius r with respect to $d(\cdot, \cdot)$ is defined as usual by

$$\overline{B}(x, r) = \overline{B}_d(x, r) = \{y \in X : d(x, y) \leq r\}. \tag{3.1}$$

One can check that this is always a closed set in X with respect to the topology determined by $d(\cdot, \cdot)$ as in the preceding section, by standard arguments. As in (2.5), we have that

$$\overline{B}_{d^q}(x, r^q) = \overline{B}_d(x, r) \tag{3.2}$$

for every $x \in X$ and $r \geq 0$. This permits one to reduce the previous statement about closed balls being closed sets to the case of an ordinary semimetric on X.

Alternatively, if $z \in X \setminus \overline{B}(x, r)$ and $t > 0$ satisfy

$$r^q + t^q \leq d(x, z)^q, \tag{3.3}$$

then one can verify that

$$B(z, t) \subseteq X \setminus \overline{B}(x, r), \tag{3.4}$$

using the q-semimetric version of the triangle inequality. Of course, if z is not in $\overline{B}(x, r)$, then $d(x, z) > r$, and so

$$t = (d(x, z)^q - r^q)^{1/q} > 0. \tag{3.5}$$

By construction, equality holds in (3.3) with this choice of t, so that (3.4) holds as well. This implies that $X \setminus \overline{B}(x, r)$ is an open set in X with respect to the topology determined by $d(\cdot, \cdot)$. Equivalently, $\overline{B}(x, r)$ is a closed set with respect to this topology, as before.

Suppose now that $d(\cdot, \cdot)$ is a semi-ultrametric on X. If $z \in X \setminus \overline{B}(x, r)$, then

$$B(z, d(x, z)) \subseteq X \setminus \overline{B}(x, r), \tag{3.6}$$

which is to say that (3.4) holds with $t = d(x, z)$. More precisely, if $x, y, z \in X$ satisfy

$$d(y, z) < d(x, z), \tag{3.7}$$

then the ultrametric version of the triangle inequality (1.9) implies that

$$d(x, z) \leq d(x, y). \tag{3.8}$$

This implies that (3.6) holds when $d(x, z) \geq r$, and indeed (3.8) is the same as (3.6) with $r = d(x, z)$. If $x, y, z \in X$ satisfy (3.7), then we also have that

$$d(x, y) \leq d(x, z) \tag{3.9}$$

by the ultrametric version of the triangle inequality, and hence

$$d(x, y) = d(x, z). \tag{3.10}$$

If $x, z \in X$ satisfy

$$d(x, z) \leq r \tag{3.11}$$

for some $r \geq 0$, then it is easy to see that

$$\overline{B}(z, r) \subseteq \overline{B}(x, r), \tag{3.12}$$

using the ultrametric version of the triangle inequality. In particular, this implies that $\overline{B}(x,r)$ is an open set in X with respect to the topology determined by $d(\cdot,\cdot)$ when $r > 0$ and $d(\cdot,\cdot)$ is a semi-ultrametric on X. It follows from (3.12) that

$$\overline{B}(x,r) = \overline{B}(z,r) \tag{3.13}$$

when (3.11) holds, by interchanging the roles of x and z in (3.12). If $x, z \in X$ satisfy

$$\overline{B}(x,r) \cap \overline{B}(z,r) \neq \emptyset \tag{3.14}$$

for some $r \geq 0$, then (3.11) holds, by the ultrametric version of the triangle inequality. Thus (3.14) implies (3.13) when $d(\cdot,\cdot)$ is a semi-ultrametric on X.

4 q-Absolute value functions

Let k be a field, and let q be a positive real number. A nonnegative real-valued function $|\cdot|$ on k is said to be a *q-absolute value function* on k if it satisfies the following three conditions. The first condition asks that

$$|x| = 0 \quad \text{if and only if} \quad x = 0. \tag{4.1}$$

The second and third conditions ask that

$$|x\,y| = |x|\,|y| \tag{4.2}$$

and

$$|x+y|^q \leq |x|^q + |y|^q \tag{4.3}$$

for every $x, y \in k$. If $|\cdot|$ satisfies these conditions on k with $q = 1$, then we simply say that $|\cdot|$ is an *absolute value function* on k.

We shall use 1 to denote both the multiplicative identity element in k and the positive real number, as appropriate. Using (4.2), we get that

$$|1|^2 = |1^2| = |1|, \tag{4.4}$$

and hence

$$|1| = 1, \tag{4.5}$$

because $|1| > 0$, by (4.1). If $x \in k$ satisfies $x^n = 1$ for some positive integer n, then it follows that

$$|x|^n = |x^n| = 1, \tag{4.6}$$

so that $|x| = 1$. In particular, this holds when $x = -1$ in k, which implies that

$$|-z| = |z| \tag{4.7}$$

for every $y \in k$. Using this, (4.1), and (4.3), it is easy to see that

$$d(x,y) = |x-y| \tag{4.8}$$

defines a q-metric on k.

If $|\cdot|$ is a nonnegative real-valued function on k that satisfies (4.1), (4.2), and

$$|x+y| \le \max(|x|, |y|) \tag{4.9}$$

for every $x, y \in k$, then $|\cdot|$ is said to be an *ultrametric absolute value function* on k. In this case, (4.8) is an ultrametric on k. The *trivial absolute value function* on k is defined by putting $|0| = 0$ and $|x| = 1$ for every $x \in k$ with $x \neq 0$. It is easy to see that this is an ultrametric absolute value function on k, for which the corresponding ultrametric as in (4.8) is the discrete metric on k. Note that an ultrametric absolute value function on k is also a q-absolute value function on k for every $q > 0$.

As usual, (4.3) is equivalent to saying that

$$|x+y| \le (|x|^q + |y|^q)^{1/q} \tag{4.10}$$

for every $x, y \in k$. If $|\cdot|$ is a q_2-absolute value function on k for some $q_2 > 0$, then $|\cdot|$ is also a q_1-absolute value function on k when $0 < q_1 \le q_2$, by (1.8). An ultrametric absolute value function may be considered as a q-absolute value function with $q = \infty$, because of (1.12). If q is any positive real number, then a nonnegative real-valued function $|\cdot|$ on k is a q-absolute value function exactly when $|\cdot|^q$ is an ordinary absolute value function on k. Of course, the standard absolute value functions on the fields $\mathbf{R}$ and $\mathbf{C}$ of real and complex numbers, respectively, are absolute value functions in this sense.

5 q-Seminorms

Let k be a field, let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let V be a vector space over k. A nonnegative real-valued function N on V is said to be a *q-seminorm* on V with respect to $|\cdot|$ on k for some positive real number q if

$$N(t\,v) = |t|\,N(v) \tag{5.1}$$

for every $t \in k$ and $v \in V$, and

$$N(v+w)^q \le N(v)^q + N(w)^q \tag{5.2}$$

for every $v, w \in V$. In particular, (5.1) implies that $N(0) = 0$, by taking $t = 0$. If we also have that

$$N(v) > 0 \tag{5.3}$$

for every $v \in V$ with $v \neq 0$, then N is said to be a *q-norm* on V. Of course, k may be considered as a 1-dimensional vector space over itself, and $|\cdot|$ may be considered as a q_k-norm on k as a vector space over itself.

Similarly, a nonnegative real-valued function N on V is said to be a *semi-ultranorm* on V with respect to $|\cdot|$ on k if N satisfies (5.1) and

$$N(v+w) \le \max(N(v), N(w)) \tag{5.4}$$

for every $v, w \in V$. As usual, one can check that (5.4) implies (5.2) for every $q > 0$, so that a semi-ultranorm on V may be considered as a q-seminorm for every $q > 0$. If a semi-ultranorm N on V satisfies (5.3) too, then N is said to be an *ultranorm* on V with respect to $|\cdot|$ on k. In this case, N may be considered as a q-norm on V for every $q > 0$, as before. If $|\cdot|$ is an ultrametric absolute value function on k, then $|\cdot|$ may be considered as an ultranorm on k as a vector space over itself, as in the preceding paragraph.

As in the previous situations, (5.2) can be reformulated as saying that

$$N(v + w) \le (N(v)^q + N(w)^q)^{1/q} \tag{5.5}$$

for every $v, w \in V$. If N is a q_2-seminorm on V with respect to $|\cdot|$ on k for some $q_2 > 0$, and if $0 < q_1 \le q_2$, then N is a q_1-seminorm on V with respect to $|\cdot|$ on k as well, because of (1.8). This implies the analogous statement for q_2-norms on V. Semi-ultranorms and ultranorms on V may be considered as q-seminorms and q-norms on V with $q = \infty$, respectively, because of (1.12).

Suppose for the moment that N is a q-seminorm on V for some $q > 0$, and that $N(v) > 0$ for some $v \in V$. Under these conditions, it is easy to see that $|\cdot|$ has to satisfy (4.3) on k with the same choice of q. This also works when $q = \infty$, in the sense that if N is a semi-ultranorm on V such that $N(v) > 0$ for some $v \in V$, then $|\cdot|$ should satisfy the ultrametric version of the triangle inequality (4.9). Thus one may wish to restrict one's attention to $q > 0$ such that $|\cdot|$ is a q-absolute value function on k.

If $q = 1$, then a q-seminorm or q-norm on V may simply be called a seminorm or a norm on V, respectively. Remember that $|\cdot|$ is a q-absolute value function on k for some positive real number q if and only if $|\cdot|^q$ is an ordinary absolute value function on k. In this case, a nonnegative real-valued function N on V is a q-seminorm with respect to $|\cdot|$ on k if and only if N^q is a seminorm on V with respect to $|\cdot|^q$ as an ordinary absolute value function on k, and similarly for q-norms.

If N is a q-seminorm on V for some $q > 0$, then

$$d(v, w) = d_N(v, w) = N(v - w) \tag{5.6}$$

defines a q-semimetric on V, which is a q-metric on V when N is a q-norm on V. This uses the fact that $|-1| = 1$, as in (4.6), to get that (5.6) is symmetric in v and w. The previous statement also works when $q = \infty$, in the sense that (5.6) is a semi-ultrametric on V when N is a semi-ultranorm on V, and that (5.6) is an ultrametric on V when N is an ultranorm on V. If $|\cdot|$ is the trivial absolute value function on k, then the *trivial ultranorm* is defined on V by putting $N(0) = 0$ and $N(v) = 1$ for every $v \in V$ with $v \ne 0$. It is easy to see that this defines an ultranorm on V, for which the corresponding ultrametric as in (5.6) is the discrete metric.

Suppose now that N is a semi-ultranorm on V, and observe that

$$N(v) \le \max(N(w), N(v - w)) \tag{5.7}$$

for every $v, w \in V$. If

(5.8) $$N(v - w) < N(v),$$

then this implies that

(5.9) $$N(v) \le N(w).$$

It is easy to see that the reverse inequality also holds when $v, w \in V$ satisfy (5.8), so that

(5.10) $$N(v) = N(w)$$

under these conditions. This is basically the same as (3.10) in this situation, using the semi-ultrametric (5.6) corresponding to N on V. In particular, if $x, y \in k$ satisfy $|x - y| < |x|$, then we get that

(5.11) $$|x| = |y|,$$

for the same reasons.

6 Lipschitz mappings

Let X, Y be sets, and suppose that d_X, d_Y are q_X, q_Y-semimetrics on X, Y, respectively, for some $q_X, q_Y > 0$. A mapping f from X into Y is said to be *Lipschitz* of order $\alpha > 0$ if there is a nonnegative real number C such that

(6.1) $$d_Y(f(x), f(x')) \le C\, d_X(x, x')^\alpha$$

for every $x, x' \in X$. Of course, constant mappings satisfy this condition with $C = 0$. If d_Y is a q_Y-metric on Y, and if f satisfies (6.1) with $C = 0$, then f is constant on X.

Let Z be another set, and let d_Z be a q_Z-semimetric on Z for some $q_Z > 0$. Suppose that $f : X \to Y$ is Lipschitz of order $\alpha > 0$ with constant $C(f) \ge 0$, and that $g : Y \to Z$ is Lipschitz of order $\beta > 0$ with constant $C(g) \ge 0$. Thus the composition $g \circ f$ of f and g is defined as a mapping from X into Z, and

(6.2) $$\begin{aligned} d_Z((g \circ f)(x'), (g \circ f)(x')) &= d_Z(g(f(x)), g(f(x'))) \\ &\le C(g)\, d_Y(f(x), f(x'))^\beta \\ &\le C(g)\, C(f)^\beta\, d_X(x, x')^{\alpha\,\beta} \end{aligned}$$

for every $x, x' \in X$. This shows that $g \circ f$ is Lipschitz of order $\alpha\,\beta$ with constant $C(f)^\beta\, C(g)$ as a mapping from X into Z under these conditions.

To simplify notation, let us now take X to be a set with a q-semimetric d for some $q > 0$. Let $x_0 \in X$ be given, and put

(6.3) $$f_0(x) = d(x, x_0)^q$$

for every $x \in X$, so that f_0 defines a real-valued function on X. Observe that

(6.4) $$f_0(x) - f_0(x') \le d(x, x')^q$$

for every $x, x' \in X$, by the q-semimetric version of the triangle inequality (1.3). Similarly,

$$f_0(x') - f_0(x) \le d(x, x')^q \tag{6.5}$$

for every $x, x' \in X$, by exchanging the roles of x and x' in (6.4). Combining (6.4) and (6.5), we get that that

$$|f_0(x) - f_0(x')| \le d(x, x')^q \tag{6.6}$$

for every $x, x' \in X$, using the standard absolute value function on the real line on the left side of this inequality. This shows that f_0 is Lipschitz of order q with constant $C = 1$ as a mapping from X into the real line, using the standard metric on $\mathbf{R}$ in the range of this mapping. If $0 < q_0 \le q$, then d is also a q_0-semimetric on X, as in Section 1. In this case, the preceding argument implies that

$$d(x, x_0)^{q_0} \tag{6.7}$$

is Lipschitz of order q_0 with constant $C = 1$ as a real-valued function of $x \in X$.

Now let $|\cdot|$ be a q-absolute value function on a field k for some $q > 0$, so that

$$|x - y| \tag{6.8}$$

defines a q-metric on k, as in (4.8). If $0 < q_0 \le q$, then we can apply the argument in the previous paragraph with $x_0 = 0$, to get that

$$|x|^{q_0} \tag{6.9}$$

is Lipschitz of order q_0 with constant $C = 1$ on k with respect to (6.8). Similarly, let V be a vector space over k, and let N be a q-seminorm on V with respect to $|\cdot|$ on k, so that

$$N(v - w) \tag{6.10}$$

is a q-semimetric on V, as in (5.6). As before, if $0 < q_0 \le q$, then we can apply the argument in the previous paragraph with $x_0 = 0$, to get that

$$N(v)^{q_0} \tag{6.11}$$

is Lipschitz of order q_0 with constant $C = 1$ on V with respect to (6.10).

7 Uniform continuity and completeness

Let X, Y be sets with q_X, q_Y-semimetrics d_X, d_Y again, respectively, and for some $q_X, q_Y > 0$. One can define uniform continuity for mappings from X into Y with respect to d_X and d_Y, in the same way as for ordinary metric spaces. As usual, one can reduce to the case where $q_X = q_Y = 1$, by replacing d_X, d_Y with their q_Xth, q_Yth powers, respectively, and this would lead to an equivalent formulation of uniform continuity. Of course, uniformly continuous mappings are continuous with respect to the topologies associated to the corresponding semimetrics, and it is easy to see that compositions of uniformly continuous mappings are uniformly continuous. Observe that Lipschitz mappings of any positive order are uniformly continuous.

One can define the notion of a Cauchy sequence of elements of a q-metric space for any $q > 0$ in the same way as for ordinary metric spaces, or reduce to that case in the usual way. Similarly, a q-metric space is said to be *complete* if every Cauchy sequence of elements of the space converges to an element of the space. Any q-metric space can be embedded isometrically onto a dense subset of a complete q-metric space, and this completion is unique up to isometric equivalence. As before, this can be derived from the analogous statement for ordinary metric spaces by replacing the q-metric on X with its qth power, or obtained using the same type of arguments. Note that the completion of an ultrametric space is an ultrametric space as well, which corresponds to taking $q = \infty$, in which case the ultrametric is already an ordinary metric.

Let X, Y be q_X, q_Y-metric spaces for some $q_X, q_Y > 0$, and let E be a dense subset of X. Suppose that f is a uniformly continuous mapping from E into Y, with respect to the restriction of the q_X-metric on X to E. If Y is complete, then there is a unique extension of f to a uniformly continuous mapping from X into Y. This is well known for ordinary metric spaces, and otherwise one can reduce to that case in the usual way, or use essentially the same arguments in this situation. More precisely, uniqueness of the extension only requires ordinary continuity, instead of uniform continuity.

Let k be a field, and let $|\cdot|$ be a q-absolute value function on k for some $q > 0$, which leads to a q-metric on k as in (4.8). If k is not already complete with respect to this q-metric, then one can pass to a completion of k as a q-metric space, as before. It is well known that addition and multiplication on k can be extended to the completion in a natural way, so that the completion becomes a field. This will be discussed further in the next paragraph. The extension of the q-metric to the completion of k includes an extension of $|\cdot|$ to the completion, since this function is the same as the distance to 0 with respect to the q-metric. This extension of $|\cdot|$ to the completion of k is a q-absolute value function on the completion, and the extension of the q-metric to the completion is the same as the q-metric associated to the extension of $|\cdot|$ to the completion. If $|\cdot|$ is an ultrametric absolute value function on k, then its extension to the completion of k is an ultrametric absolute value function too.

More precisely, it is easy to see that addition on k is continuous as a mapping from $k \times k$ into k, using the corresponding product topology on $k \times k$. In fact, addition on k is uniformly continuous as a mapping from $k \times k$ into k, with respect to the q-metric on $k \times k$ obtained by taking the maximum of the distances in the two coordinates. The extension of addition to the completion of k can be obtained from this uniform continuity as before. Similarly, multiplication on k is continuous as a mapping from $k \times k$ into k, and it is uniformly continuous on products of balls in k. This can be used to extend multiplication to the completion of k. The mapping

$$x \mapsto 1/x \tag{7.1}$$

is continuous on $k \setminus \{0\}$, and uniformly continuous on the set of $x \in k$ with

$|x| \geq c$ for any $c > 0$. This permits one to extend (7.1) to nonzero elements of the completion of k.

Let V be a vector space over k, and let N be a q-norm on V, so that (5.6) is a q-metric on V. If V is not already complete as a q-metric space, then one can pass to a completion, as before. Addition and scalar multiplication can be extended to the completion in a natural way, so that the completion is also a vector space over k. The extension of addition can be obtained from the uniform continuity of addition on V as a mapping from $V \times V$ into V, using the q-metric on $V \times V$ given by the maximum of the distances in the two coordinates. To extend scalar multiplication to the completion of V, one can use the uniform continuity of

$$v \mapsto t\, v \tag{7.2}$$

as a mapping from V into itself for each $t \in k$. The distance to 0 in the completion of V defines an extension of N to the completion of V, and this extension of N is a q-norm on the completion of V, which corresponds to the extension of the q-metric to the completion of V in the usual way. If N is an ultranorm on V, then the extension of N to the completion of V is an ultranorm.

If V is complete with respect to the q-metric associated to N, but k is not complete with respect to the q-metric associated to $|\cdot|$, then one can extend scalar multiplication on V to the completion of k in a natural way. This can be obtained from the uniform continuity of

$$t \mapsto t\, v \tag{7.3}$$

as a mapping from k into V for each $v \in V$. Using this extension of scalar multiplication, V becomes a vector space over the completion of k. Similarly, N is a q-norm on V as a vector space over the completion of k, and with respect to the extension of $|\cdot|$ to the completion of k discussed earlier. This includes the case where $q = \infty$, which corresponds to ultranorms and ultrametric absolute value functions.

8 Infinite series

Let k be a field, and let $|\cdot|$ be a q-absolute value function on k for some $q > 0$. Also let V be a vector space over k, and let N be a q-norm on V with respect to $|\cdot|$ on k. As usual, an infinite series $\sum_{j=1}^\infty v_j$ with terms in V is said to *converge* in V if the corresponding sequence of partial sums

$$\sum_{j=1}^{l} v_j \tag{8.1}$$

converges to an element of V with respect to the q-metric (5.6) corresponding to N. In this case, the value of the sum $\sum_{j=1}^\infty v_j$ is defined to be the limit of the sequence of partial sums (8.1) in V. In particular, the partial sums (8.1) are bounded in V with respect to N when they converge, and one can check that

$$N\Big(\sum_{j=1}^\infty v_j\Big) \le \sup_{l \ge 1} N\Big(\sum_{j=1}^l v_j\Big). \tag{8.2}$$

If $\sum_{j=1}^\infty v_j$ converges in V and $t \in k$, then $\sum_{j=1}^\infty t\, v_j$ converges in V, and

$$\sum_{j=1}^\infty t\, v_j = t \sum_{j=1}^\infty v_j, \tag{8.3}$$

by continuity of scalar multiplication on V. Similarly, if $\sum_{j=1}^\infty v_j$, $\sum_{j=1}^\infty w_j$ are convergent infinite series with terms in V, then $\sum_{j=1}^\infty (v_j + w_j)$ converges in V, and

$$\sum_{j=1}^\infty (v_j + w_j) = \sum_{j=1}^\infty v_j + \sum_{j=1}^\infty w_j, \tag{8.4}$$

by continuity of addition on V.

Let $\sum_{j=1}^\infty v_j$ be an infinite series with terms in V again. The sequence of partial sums (8.1) is a Cauchy sequence in V with respect to the q-metric associated to N if and only if for each $\epsilon > 0$ there is a positive integer $L = L(\epsilon)$ such that

$$N\Big(\sum_{j=l}^r v_j\Big) < \epsilon \tag{8.5}$$

for every $r \ge l \ge L$. This implies in particular that

$$\lim_{j \to \infty} N(v_j) = 0, \tag{8.6}$$

by taking $l = r$ in (8.5). Of course, if $\sum_{j=1}^\infty v_j$ converges in V, then the sequence of partial sums (8.1) is a Cauchy sequence in V.

Let us say that $\sum_{j=1}^\infty v_j$ converges *q-absolutely* if $q < \infty$ and

$$\sum_{j=1}^\infty N(v_j)^q \tag{8.7}$$

converges as an infinite series of nonnegative real numbers. The q-norm version (5.2) of the triangle inequality implies that

$$N\Big(\sum_{j=l}^r v_j\Big)^q \le \sum_{j=l}^r N(v_j)^q \tag{8.8}$$

for every $r \ge l \ge 1$. If $\sum_{j=1}^\infty v_j$ converges q-absolutely, then it follows that the sequence of partial sums (8.1) is a Cauchy sequence in V with respect to N.

If V is complete with respect to the q-metric associated to N, then this means that $\sum_{j=1}^{\infty} v_j$ converges in V. In this case, we get that

$$N\Big(\sum_{j=1}^{\infty} v_j\Big)^q \le \sum_{j=1}^{\infty} N(v_j)^q, \tag{8.9}$$

by combining (8.2) and (8.8).

Suppose now that N is an ultranorm on V, which corresponds to $q = \infty$. Thus

$$N\Big(\sum_{j=l}^{r} v_j\Big) \le \max_{l \le j \le r} N(v_j) \tag{8.10}$$

for every $r \ge l \ge 1$, by the ultranorm version (5.4) of the triangle inequality. If $\{v_j\}_{j=1}^{\infty}$ converges to 0 in V with respect to N, as in (8.6), then the sequence of partial sums (8.1) is a Cauchy sequence in V with respect to N. This implies that $\sum_{j=1}^{\infty} v_j$ converges in V if V is also complete with respect to the ultrametric associated to N. Under these conditions, we have that

$$N\Big(\sum_{j=1}^{\infty} v_j\Big) \le \max_{j \ge 1} N(v_j), \tag{8.11}$$

by combining (8.2) and (8.10). More precisely, the existence of the maximum on the right side of (8.11) is trivial when $v_j = 0$ for every j. Otherwise, if $v_j \ne 0$ for some j, and hence $N(v_j) > 0$ for some j, then the right side of (8.11) reduces to the maximum of finitely many terms, because of (8.6).

9 Bounded sets and mappings

Let Y be a set, and let $d(\cdot, \cdot)$ be a q-semimetric on Y for some $q > 0$. A subset E of Y is said to be *bounded* with respect to $d(\cdot, \cdot)$ if

$$\{d(y, z) : y, z \in E\} \tag{9.1}$$

has a finite upper bound in $\mathbf{R}$. If E is bounded and w is any element of Y, then it is easy to see that E is contained in a ball of finite radius about w with respect to $d(\cdot, \cdot)$. Conversely, if E is contained in an open or closed ball of finite radius in Y with respect to $d(\cdot, \cdot)$, then E is bounded in Y. Note that the union of finitely many bounded subsets of Y with respect to $d(\cdot, \cdot)$ is also bounded with respect to $d(\cdot, \cdot)$.

A mapping f from a set X into Y is said to be *bounded* with respect to $d(\cdot, \cdot)$ if $f(X)$ is bounded as a subset of Y with respect to $d(\cdot, \cdot)$. Let $\mathcal{B}(X, Y) = \mathcal{B}_d(X, Y)$ be the space of bounded mappings from X into Y with respect to $d(\cdot, \cdot)$. If f, g are bounded mappings from X into Y, then

$$d(f(x), g(x)) \tag{9.2}$$

is bounded as a nonnegative real-valued function on X. This implies that the supremum

$$\sup_{x \in X} d(f(x), g(x)) \tag{9.3}$$

is defined as a nonnegative real number when $X \neq \emptyset$. One can check that (9.3) defines a q-semimetric on $\mathcal{B}(X, Y)$, which is the *supremum q-semimetric* associated to $d(\cdot, \cdot)$ on Y. If $d(\cdot, \cdot)$ is a q-metric on Y, then (9.3) is a q-metric on $\mathcal{B}(X, Y)$, and may be called the supremum q-metric associated to $d(\cdot, \cdot)$ on Y. There are analogous statements for semi-ultrametrics and ultrametrics, which correspond to $q = \infty$.

Suppose that $d(\cdot, \cdot)$ is a q-metric on Y, and that Y is complete as a q-metric space with respect to $d(\cdot, \cdot)$. If X is any nonempty set, then $\mathcal{B}(X, Y)$ is complete with respect to the corresponding supremum q-metric (9.3). This is well known when $q = 1$, and otherwise one can reduce to this case in the usual way, or use essentially the same arguments as wehn $q = 1$. More precisely, if $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence in $\mathcal{B}(X, Y)$ with respect to the supremum q-metric, then $\{f_j(x)\}_{j=1}^\infty$ is a Cauchy sequence in Y for every $x \in X$. If Y is complete, then it follows that for each $x \in X$, there is an element $f(x)$ of Y such that $\{f_j(x)\}_{j=1}^\infty$ converges to $f(x)$ with respect to $d(\cdot, \cdot)$ on Y. One can check that f is bounded as a mapping from X into Y, because $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to the supremum q-metric, and hence bounded with respect to the supremum q-metric. Similarly, one can use the Cauchy condition for $\{f_j\}_{j=1}^\infty$ again to show that this sequence converges to f with respect to the supremum q-metric.

Let k be a field with a q-absolute value function $|\cdot|$ for some $q > 0$, and let V be a vector space over k with a q-seminorm N with respect to $|\cdot|$ on k. Thus N determines a q-semimetric on V, as in (5.6). Let X be a nonempty set, and observe that a V-valued function f on X is bounded with respect to this q-semimetric if and only if $N(f(x))$ is bounded as a nonnegative real-valued function on X. In this case, the space of bounded V-valued functions on X may also be denoted $\ell^\infty(X, V)$, and it is easy to see that this is a vector space over k with respect to pointwise addition and scalar multiplication. If f is a bounded V-valued function on X, then the corresponding *supremum q-seminorm* is defined by

$$\|f\|_\infty = \|f\|_{\ell^\infty(X,V)} = \sup_{x \in X} N(f(x)). \tag{9.4}$$

One can verify that this is a q-seminorm on $\ell^\infty(X, V)$ under these conditions, for which the corresponding q-semimetric is the same as the supremum q-semimetric associated to the q-semimetric on V determined by N. If N is a q-norm on V, then (9.4) is a q-norm on $\ell^\infty(X, V)$, which may be called the supremum q-norm on $\ell^\infty(X, V)$ associated to N on V. As before, there are analogous statements for semi-ultranorms and ultranorms, which correspond to $q = \infty$.

10 r-Summable functions

Let X be a nonempty set, and let f be a nonnegative real-valued function defined on X. Of course, if X has only finitely many elements, then the sum

$$\sum_{x \in X} f(x) \tag{10.1}$$

can be defined in the usual way. Otherwise, (10.1) can be defined as a nonnegative extended real number as the supremum of

$$\sum_{x \in A} f(x) \tag{10.2}$$

over all finite subsets A of X. Thus (10.1) is finite if and only if there is a finite upper bound for the finite subsums (10.2), in which case f is said to be *summable* on X. If g is another nonnegative real-valued function on X, then

$$\sum_{x \in X} (f(x) + g(x)) = \sum_{x \in X} f(x) + \sum_{x \in X} g(x), \tag{10.3}$$

with the usual conventions for sums of nonnegative extended real numbers. Similarly, if a is a positive real number, then

$$\sum_{x \in X} a\, f(x) = a \sum_{x \in X} f(x), \tag{10.4}$$

where the right side is interpreted as being $+\infty$ when (10.1) is infinite. If $a = 0$, then the right side of (10.4) should be interpreted as being equal to 0, even when (10.1) is infinite. If X is the set $\mathbf{Z}_+$ of positive integers, then this definition of (10.1) is equivalent to taking the supremum of the partial sums from 1 to n.

Let k be a field with a q-absolute value function for some $q > 0$, let V be a vector space over k with a q-seminorm N with respect to $|\cdot|$ on k, and let r be a positive real number. A V-valued function f on X is said to be *r-summable* on X with respect to N if $N(f(x))^r$ is summable as a nonnegative real-valued function on X. Let $\ell^r(X, V)$ be the space of r-summable V-valued functions on X with respect to N. If f is such a function, then we put

$$\|f\|_r = \|f\|_{\ell^r(X,V)} = \Big(\sum_{x \in X} N(f(x))^r\Big)^{1/r}, \tag{10.5}$$

and otherwise we may interpret the right side as being $+\infty$ when $N(f(x))^r$ is not summable on X. Observe that

$$\|t\, f\|_r = |t|\, \|f\|_r \tag{10.6}$$

for every $t \in k$ and V-valued function f on X, with the usual interpretations when f is not r-summable on V with respect to N.

If f, g are V-valued functions on X, then

$$\|f+g\|_r^r \le \|f\|_r^r + \|g\|_r^r \tag{10.7}$$

when $0 < r \le q$, and

$$\|f+g\|_r^q \le \|f\|_r^q + \|g\|_r^q \tag{10.8}$$

when $r \ge q$. Of course, (10.7) and (10.8) are the same when $r = q$, in which case one can simply use the q-seminorm version of the triangle inequality for N. If $r \le q$, then N is also an r-seminorm on V, as in Section 5, and so one can use the same argument as when $r = q$. If $r \ge q$, then (10.8) can be verified using Minkowski's inequality for sums, corresponding to the exponent $r/q \ge 1$. Both (10.7) and (10.8) are trivial unless f and g are r-summable on X with respect to N, in which case $f + g$ is r-summable on X as well. It follows that $\ell^r(X, V)$ is a vector space with respect to pointwise addition and scalar multiplication for every $r > 0$, and that (10.5) defines a q-seminorm on $\ell^r(X, V)$ when $r \ge q$, and an r-seminorm when $r \le q$. If N is a q-norm on V, then (10.5) is a q or r-norm on $\ell^r(X, V)$, as appropriate.

If a V-valued function f on X is r-summable with respect to N for some $r > 0$, then f is bounded with respect to N too, and

$$\|f\|_\infty \le \|f\|_r. \tag{10.9}$$

Suppose that N is a q-norm on V, and that V is complete with respect to the corresponding q-metric, so that $\ell^\infty(X, V)$ is complete with respect to the corresponding supremum q-metric, as in the previous section. Under these conditions, $\ell^r(X, V)$ is complete with respect to the q or r-metric associated to (10.5), as appropriate. Of course, any Cauchy sequence in $\ell^r(X, V)$ is a Cauchy sequence in $\ell^\infty(X, V)$ as well, because of (10.9), and hence converges in $\ell^\infty(X, V)$. To show that $\ell^r(X, V)$ is complete, one should check that the limit of the sequence is r-summable on X with respect to N, and that the sequence converges with respect to (10.5).

11 Vanishing at infinity

Let k be a field with a q-absolute value function for some $q > 0$, let V be a vector space over k with a q-seminorm N with respect to $|\cdot|$ on k, and let X be a nonempty set. A V-valued function f on X is said to *vanish at infinity* on X with respect to N if for each $\epsilon > 0$ there are only finitely many $x \in X$ such that

$$N(f(x)) \ge \epsilon. \tag{11.1}$$

Let $c_0(X, V)$ be the space of these functions on X. It is easy to see that $c_0(X, V)$ is a linear subspace of $\ell^\infty(X, V)$, and that $c_0(X, V)$ is a closed set in $\ell^\infty(X, V)$ with respect to the corresponding supremum q-semimetric. If X has only finitely many elements, then every V-valued function on X automatically vanishes at infinity on X. If X is the set $\mathbf{Z}_+$ of positive integers, then a V-valued function on X is the same as a sequence of elements of V. In this case, vanishing at infinity on X corresponds to the sequence converging to 0 in V with respect to N.

The *support* of a V-valued function f on X is defined to be the set of $x \in X$ such that $f(x) \neq 0$. Let $c_{00}(X, V)$ be the space of V-valued functions on X whose support has only finitely many elements. This is a linear subspace of $c_0(X, V)$, and in fact $c_{00}(X, V)$ is dense in $c_0(X, V)$ with respect to the supremum q-semimetric associated to N. If f is a V-valued function on X that vanishes at infinity on X with respect to N, then the support of $N(f(x))$ has only finitely or countably many elements. If N is a q-norm on V, then it follows that the support of f has only finitely or countably many elements.

If a V-valued function f on X is r-summable with respect to N for some positive real number r, then f vanishes at infinity on X with respect to N, so that

$$\ell^r(X, V) \subseteq c_0(X, V). \tag{11.2}$$

Of course, if a V-valued function f on X has finite support, then f is r-summable with respect to N for every $r > 0$, which is to say that

$$c_{00}(X, V) \subseteq \ell^r(X, V). \tag{11.3}$$

One can check that $c_{00}(X, V)$ is dense in $\ell^r(X, V)$ for every positive real number r, with respect to the q or r-seminorm associated to (10.5), as appropriate. If f, g are r-summable V-valued functions on X with respect to N for some $r > 0$, and if the supports of f, g are disjoint in X, then

$$\|f + g\|_r^r = \|f\|_r^r + \|g\|_r^r. \tag{11.4}$$

Similarly, if f, g are bounded V-valued functions on X with respect to N, and if the supports of f, g are disjoint in X, then

$$\|f + g\|_\infty = \max(\|f\|_\infty, \|g\|_\infty), \tag{11.5}$$

which is the analogue of (11.4) for $r = \infty$.

Let r_1, r_2 be positive real numbers with $r_1 \leq r_2$, and let f be a V-valued function on X that is r_1-summable with respect to N. This implies that f is bounded with respect to N, as in the previous section. Thus

$$\sum_{x \in X} N(f(x))^{r_2} \leq \|f\|_\infty^{r_2 - r_1} \sum_{x \in X} N(f(x))^{r_1} < \infty, \tag{11.6}$$

so that f is r_2-summable on X with respect to N as well. It follows that

$$\ell^{r_1}(X, V) \subseteq \ell^{r_2}(X, V). \tag{11.7}$$

We also get that

$$\|f\|_{r_2}^{r_2} \leq \|f\|_\infty^{r_2 - r_1} \|f\|_{r_1}^{r_1} \leq \|f\|_{r_1}^{r_2}, \tag{11.8}$$

using (11.6) in the first step, and (10.9) in the second step. Hence

$$\|f\|_{r_2} \leq \|f\|_{r_1}. \tag{11.9}$$

Note that (11.9) reduces to (1.8) when X has exactly two elements.

12 q-Absolute value functions, continued

Let k be a field, and let $|\cdot|$ be a q-absolute value function on k for some $q > 0$. If $x \in k$ and n is a positive integer, then let $n \cdot x$ denote the sum of n x's in k. If there are $n \in \mathbf{Z}_+$ such that $|n \cdot 1|$ is arbitrarily large, then $|\cdot|$ is said to be *archimedean* on k. Otherwise, $|\cdot|$ is said to be *non-archimedean* on k. Observe that

$$|n^j \cdot 1| = |(n \cdot 1)^j| = |n \cdot 1|^j \tag{12.1}$$

for all $j, n \in \mathbf{Z}_+$, and that (12.1) tends to infinity as $j \to \infty$ when $|n \cdot 1| > 1$. It follows that

$$|n \cdot 1| \le 1 \tag{12.2}$$

for every $n \in \mathbf{Z}_+$ when $|\cdot|$ is non-archimedean on k. If $|\cdot|$ is an ultrametric absolute value function on k, then it is easy to see that (12.2) holds for every $n \in \mathbf{Z}_+$, so that $|\cdot|$ is non-archimedean on k. Conversely, if a q-absolute value function $|\cdot|$ is non-archimedean on k, then it is well known that $|\cdot|$ is an ultrametric absolute value function on k.

If $|\cdot|$ is any q-absolute value function on a field k again, then the set of positive values of $|\cdot|$ on k is a subgroup of the group $\mathbf{R}_+$ of positive real numbers with respect to multiplication. If 1 is not a limit point of this set with respect to the standard topology on $\mathbf{R}$, then $|\cdot|$ is said to be *discrete* on k. In this case, one can show that the set of positive values of $|\cdot|$ on k is the same as the set of integer powers of a single positive real number. If $|\cdot|$ is not discrete on k, then one can check that the set of positive values of $|\cdot|$ on k is dense in $\mathbf{R}_+$ with respect to the standard topology on $\mathbf{R}$.

If k has positive characteristic, then any q-absolute value function on k is non-archimedean, and hence an ultrametric absolute value function. Suppose that k has characteristic 0, so that there is a natural embedding from the field $\mathbf{Q}$ of rational numbers into k. If $|\cdot|$ is a q-absolute value function on k for some $q > 0$, then this embedding induces a q-absolute value function on $\mathbf{Q}$. Note that $|\cdot|$ is archimedean on k if and only if the induced q-absolute value function on $\mathbf{Q}$ is archimedean. A famous theorem of Ostrowski implies that the only archimedean q-absolute value functions on $\mathbf{Q}$ are given by positive powers of the standard absolute value function on $\mathbf{Q}$. In particular, these absolute value functions on $\mathbf{Q}$ are not discrete, which implies that any archimedean q-absolute value function on a field k is not discrete. Equivalently, any discrete q-absolute value function on a field k is non-archimedean, and hence an ultrametric absolute value function.

Let $|\cdot|$ be an archimedean q-absolute value function on a field k again, for some $q > 0$. Suppose that k is also complete with respect to the q-metric associated to $|\cdot|$ as in (4.8). Under these conditions, another famous theorem of Ostrowski implies that k is isomorphic as a field to $\mathbf{R}$ or $\mathbf{C}$, in such a way that $|\cdot|$ corresponds to a positive power of the standard absolute value function on $\mathbf{R}$ or $\mathbf{C}$, as appropriate, using this isomorphism. Of course, if k is not complete with respect to the q-metric associated to $|\cdot|$, then one can simply pass to a completion, as in Section 7.

13 Bounded linear mappings

Let k be a field, let $|\cdot|$ be a q-absolute value function on k for some $q > 0$, and let V, W be vector spaces over k. Suppose that N_V, N_W are q_V, q_W-seminorms on V, W, respectively, and for some $q_V, q_W > 0$. One may as well ask that $q_V, q_W \le q$ here, since $|\cdot|$ should be a q_V, q_W-absolute value function on k, unless N_V or N_W is identically 0. A linear mapping T from V into W is said to be *bounded* with respect to N_V, N_W if there is a nonnegative real number C such that

$$N_W(T(v)) \le C\, N_V(v) \tag{13.1}$$

for every $v \in V$. Of course, this implies that

$$N_W(T(v) - T(v')) = N_W(T(v - v')) \le C\, N_V(v - v') \tag{13.2}$$

for every $v, v' \in V$, so that T is Lipschitz of order 1 with respect to the q_V, q_W-semimetrics on V, W associated to N_V, N_W as in (5.6), respectively. In particular, this means that T is uniformly continuous, and hence continuous. If $|\cdot|$ is nontrivial on k, and if a linear mapping T from V into W is continuous at 0, then it is easy to see that T is bounded. More precisely, it suffices that T be bounded with respect to N_W on a ball of positive radius in V with respect to N_V.

Let T be a bounded linear mapping from V into W, and put

$$\|T\|_{op} = \|T\|_{op,VW} = \inf\{C \ge 0 :\ (13.1) \text{ holds}\}, \tag{13.3}$$

which is to say that one takes the infimum of all nonnegative real numbers C such that (13.1) holds for every $v \in V$. It is easy to see that the infimum has the same property, so that (13.1) holds with $C = \|T\|_{op}$. Note that the boundedness of T implies that

$$N_W(T(v)) = 0 \tag{13.4}$$

for every $v \in V$ such that $N_V(v) = 0$. If $N_V(v) > 0$ for some $v \in V$, then

$$\|T\|_{op} = \sup\left\{\frac{N_W(T(v))}{N_V(v)} : v \in V,\ N_V(v) > 0\right\}. \tag{13.5}$$

Otherwise, $\|T\|_{op} = 0$ when $N_V(v) = 0$ for every $v \in V$.

Let $\mathcal{BL}(V, W)$ be the space of bounded linear mappings from V into W, with respect to N_V, N_W, as before. One can check that $\mathcal{BL}(V, W)$ is a vector space over k, and that $\|T\|_{op}$ is a q_W-seminorm on $\mathcal{BL}(V, W)$, because N_W is a q_W-seminorm on W. If N_W is a q_W-norm on W, then $\|T\|_{op}$ is a q_W-norm on $\mathcal{BL}(V, W)$. In this case, if W is complete with respect to the q_W-metric associated to N_W, then $\mathcal{BL}(V, W)$ is complete with respect to the q-metric associated to $\|\cdot\|_{op}$. We may also use the notation $\mathcal{BL}(V)$ for the space of bounded linear mappings from V into itself, with respect to the same q_V-seminorm N_V on both the domain and range.

Let Z be another vector space over k, and suppose that N_Z is a q_Z-seminorm on Z for some $q_Z > 0$. If T_1 is a bounded linear mapping from V into W, and

T_2 is a bounded linear mapping from W into Z, then their composition $T_2 \circ T_1$ is a bounded linear mapping from V into Z, and

$$\|T_2 \circ T_1\|_{op,VZ} \le \|T_1\|_{op,VW} \, \|T_2\|_{op,WZ}. \tag{13.6}$$

In particular, $\mathcal{BL}(V)$ is an algebra with respect to composition of linear mappings. The identity mapping $I = I_V$ on V is bounded as a linear mapping from V into itself, with

$$\|I_V\|_{op,VV} = 1 \tag{13.7}$$

as long as $N_V(v) > 0$ for some $v \in V$.

14 r-Summable functions, continued

Let k be a field with a q-absolute value function $|\cdot|$ for some $q > 0$, and let V, W be vector spaces over k with q_V, q_W-norms N_V, N_W, respectively. Also let V_0 be a linear subspace of V, and let T be a bounded linear mapping from V_0 into W, using the restriction of N_V to V_0. Thus T is uniformly continuous on V_0, as in the previous section. If V_0 is dense in V with respect to the q_V-metric associated to N_V, and if W is complete with respect to the q_W-metric associated to N_W, then there is a unique extension of T to a uniformly continuous mapping from V into W, as in Section 7. In this situation, one can check that this extension of T to V is a bounded linear mapping from V into W, and with the same operator norm on V as on V_0.

Let k be a field again, and let V be a vector space over k. Also let X be a nonempty set, and let $c_{00}(X, V)$ be the space of V-valued functions on X with finite support, as in Section 11. If f is such a function, then the sum

$$\sum_{x \in X} f(x) \tag{14.1}$$

can be defined as an element of V in a natural way, by reducing to a finite sum. This defines a linear mapping from $c_{00}(X, V)$ into V. Let $|\cdot|$ be a q-absolute value function on k for some $q > 0$, and let N be a q-seminorm on V. Using the q-seminorm version of the triangle inequality, it is easy to see that

$$N\Big(\sum_{x \in X} f(x)\Big) \le \|f\|_{\ell^q(X,V)} \tag{14.2}$$

for every $f \in c_{00}(X, V)$. Here $\|f\|_{\ell^q(X,V)}$ is as in (10.5) when $0 < q < \infty$, and is as in (9.4) when $q = \infty$. Thus the mapping from f to the sum (14.1) defines a bounded linear mapping from $c_{00}(X, V)$ into V, using the q-seminorm $\|f\|_{\ell^q(X,V)}$ on $c_{00}(X, V)$ and N on V.

Suppose now that N is a q-norm on V, and that V is complete with respect to the corresponding q-metric. If $0 < q < \infty$, then the mapping from $f \in c_{00}(X, V)$ to the sum (14.2) can be extended to a bounded linear mapping from $\ell^q(X, V)$ into V, by the remarks at the beginning of the section. This uses the fact that

$c_{00}(X, V)$ is dense in $\ell^q(X, V)$ when $q < \infty$, as in Section 11. Similarly, if $q = \infty$, then the mapping from $f \in c_{00}(X, V)$ to the sum (14.1) can be extended to a bounded linear mapping from $c_0(X, V)$ into V, using the supremum norm on $c_0(X, V)$ corresponding to N on V, as before. This uses the fact that $c_{00}(X, V)$ is dense in $c_0(X, V)$ with respect to the supremum norm.

Of course, if X has only finitely many elements, then the sum (14.1) is already defined for all V-valued functions f on X. Otherwise, if X has infinitely many elements, then the extensions discussed in the preceding paragraph can be used to define (14.1) for $f \in \ell^q(X, V)$ when $q < \infty$, and for $f \in c_0(X, V)$ when $q = \infty$, under the conditions mentioned earlier. In both cases, (14.2) still holds for all such f. If X is the set $\mathbf{Z}_+$ of positive integers, then the sum (14.1) can also be considered as an infinite series, as in Section 8. More precisely, this corresponds to q-absolute convergence when $q < \infty$, and to infinite series with terms converging to 0 when $q = \infty$. One can check that these two approaches to defining the sum (14.1) are equivalent under these conditions, by approximating the sum by finite sums. If X is any set and f is a V-valued function that vanishes at infinity on X, then the support of f has only finitely or countably many elements, as in Section 11. This permits the sum (14.1) to either be reduced to a finite sum, or to be considered as an infinite series. Remember that q-summable functions on X vanish at infinity when $q < \infty$, so that the previous remark applies to all $q > 0$.

15 Mappings on $c_{00}(X, k)$

Let k be a field, and let X be a nonempty set. If $y \in X$, then let $\delta_y(x)$ be the k-valued function on X which is equal to 1 when $x = y$, and to 0 otherwise. Thus $\delta_y \in c_{00}(X, k)$ for every $y \in X$, and the collection of these functions forms a basis for $c_{00}(X, k)$ as a vector space over k. Let V be a vector space over k, and let a be a V-valued function on X. If $f \in c_{00}(X, k)$, then put

$$T_a(f) = \sum_{y \in X} a(y)\, f(y), \tag{15.1}$$

where the sum on the right reduces to a finite sum in V. This defines a linear mapping from $c_{00}(X, k)$ into V, and every linear mapping from $c_{00}(X, k)$ into V is of this form. Equivalently, T_a is the unique linear mapping from $c_{00}(X, k)$ into V such that

$$T_a(\delta_y) = a(y) \tag{15.2}$$

for every $y \in X$.

Let $|\cdot|$ be a q-absolute value function on k for some $q > 0$, which may be considered as a q-norm on k as a one-dimensional vector space over itself. This leads to the supremum q-norm $\|f\|_\infty$ on $\ell^\infty(X, k)$ as in (9.4), and to the q or r-norm $\|f\|_r$ on $\ell^r(X, k)$ as in (10.5) when $0 < r < \infty$, depending on whether $r \geq q$ or $r \leq q$. Note that

$$\|\delta_y\|_r = 1 \tag{15.3}$$

for every $y \in X$ and $r > 0$, where δ_y is as defined in the preceding paragraph. Let a be a V-valued function on X, and let T_a be the corresponding linear mapping from $c_{00}(X, k)$ into V, as in (15.1). Suppose for the moment that T_a is bounded with respect to $\|f\|_r$ on $c_{00}(X, k)$ for some $r > 0$ and a q-seminorm N on V. This means that there is a nonnegative real number C such that

$$N(T_a(f)) \le C \, \|f\|_r \tag{15.4}$$

for every $f \in c_{00}(X, k)$, as in (13.1). In particular, this implies that

$$N(a(y)) \le C \tag{15.5}$$

for every $y \in X$, because of (15.2) and (15.3).

Suppose now that a is a V-valued function on X that is bounded with respect to N on V, so that (15.5) holds for some $C \ge 0$ and every $y \in X$. Of course, if X has only finitely many elements, then every V-valued function on X is bounded with respect to N on V. It follows from (15.5) that

$$N(T_a(f)) \le C \, \|f\|_q \tag{15.6}$$

for every $f \in c_{00}(X, V)$, using the q-semimetric version of the triangle inequality. This could also be derived from (14.2), applied to

$$a \, f \tag{15.7}$$

as a V-valued function on X. As before, (15.6) says that T_a is bounded as a linear mapping from $c_{00}(X, k)$ into V, with respect to the ℓ^q q-norm on $c_{00}(X, k)$, and N on V. Suppose in addition that N is a q-norm on V, and that V is complete with respect to the corresponding q-metric on V. If $q < \infty$, then T_a can be extended to a bounded linear mapping from $\ell^q(X, k)$ into V, using the remarks at the beginning of the previous section. Similarly, if $q = \infty$, then T_a can be extended to a bounded linear mapping from $c_0(X, k)$ into V, using the supremum norm on $c_0(X, k)$.

Alternatively, if a is any V-valued function on X, then

$$f \mapsto a \, f \tag{15.8}$$

defines a linear mapping from $c_{00}(X, k)$ into $c_{00}(X, V)$, and T_a is the same as the composition of this mapping with the mapping from $c_{00}(X, V)$ into V defined by summing over X. If a is bounded on X with respect to N on V, then (15.8) defines a bounded linear mapping from $\ell^r(X, k)$ into $\ell^r(X, V)$ for every $r > 0$, and this mapping sends $c_0(X, k)$ into $c_0(X, V)$. Thus the boundedness of T_a with respect to the ℓ^q q-norm on $c_{00}(X, V)$ follows from the boundedness of (15.8) with respect to the ℓ^q q-norms on $c_{00}(X, k)$ and $c_{00}(X, V)$, and the boundedness of the mapping from $c_{00}(X, V)$ into V defined by summation over X with respect to the ℓ^q q-norm on $c_{00}(X, V)$, as in the previous section. If N is a q-norm on V, V is complete with respect to the corresponding q-metric, and $q < \infty$, then the extension of T_a to a bounded linear mapping from $\ell^q(X, k)$ into V can be obtained by composing (15.8) as a bounded linear mapping from $\ell^q(X,k)$ into $\ell^q(X,V)$ with the mapping from $\ell^q(X,V)$ into V defined by summation over X, as before. Similarly, if $q = \infty$, then the extension of T_a to a bounded linear mapping from $c_0(X,k)$ with the supremum norm into V can be obtained by composing (15.8) as a bounded linear mapping from $c_0(X,k)$ into $c_0(X,V)$ with respect to the supremum norm with the mapping from $c_0(X,V)$ into V defined by summation over X.

16 q-Semimetrification

Let X be a set, and let $d(\cdot,\cdot)$ be a nonnegative real-valued function defined on $X \times X$ that satisfies (1.1) and (1.2). If $z_1, \ldots, z_n$ is a finite sequence of elements of X and q is a positive real number, then let us define the (d,q)*-length* of $z_1, \ldots, z_n$ to be

$$\Big(\sum_{j=1}^{n-1} d(z_j, z_{j+1})^q\Big)^{1/q}. \tag{16.1}$$

As usual, we can extend this to $q = \infty$ by taking

$$\max_{1 \le j \le n-1} d(z_j, z_{j+1}). \tag{16.2}$$

Both (16.1) and (16.2) should be interpreted as being 0 when $n = 1$, and one can also simply restrict one's attention to $n \ge 2$ here. Clearly (16.2) is less than or equal to (16.1) for every $q > 0$. We also have that (16.1) is monotonically decreasing in q, as in (11.9). If $d(\cdot,\cdot)$ is a q-semimetric on X for some $q > 0$, then

$$d(z_1, z_n) \le (d,q)\text{-length of } z_1, \ldots, z_n \tag{16.3}$$

for every finite sequence of elements $z_1, \ldots, z_n$ of X, by the q-semimetric version of the triangle inequality.

If $x, x' \in X$ and $0 < q \le \infty$, then we put

$$\begin{aligned} d_q(x,x') \quad = \quad & \inf\{(d,q)\text{-lengths of finite sequences } z_1, \ldots, z_n \\ & \qquad \text{of elements of } X \text{ such that } z_1 = x \text{ and } z_n = x'\}. \end{aligned} \tag{16.4}$$

Thus

$$0 \le d_q(x,x') \le d(x,x'), \tag{16.5}$$

since we can take $n = 2$, $z_1 = x$, and $z_2 = x'$ in the infimum on the right side of (16.4). In particular, (16.4) is equal to 0 when $x = x'$, in which case one could also take $n = 1$. It is easy to see that (16.4) is symmetric in x and x', because $d(\cdot,\cdot)$ is symmetric by hypothesis. By construction, (16.4) satisfies the q-semimetric version of the triangle inequality, and hence defines a q-semimetric on X. Note that (16.4) is monotonically decreasing in q, because of the corresponding property of (d,q)-lengths of finite sequences of elements of X. If $d(\cdot,\cdot)$ is a q-semimetric on X for some $q > 0$, then

$$d(x,x') \le d_q(x,x') \tag{16.6}$$

for every $x, x' \in X$, because of (16.3). Combining this with (16.5), we get that

$$d_q(x, x') = d(x, x') \tag{16.7}$$

for every $x, x' \in X$ in this case.

Let $\rho(\cdot, \cdot)$ be another nonnegative real-valued function on $X \times X$ that vanishes on the diagonal and is symmetric, as in (1.1) and (1.2). Thus the (ρ, q)-length of any finite sequence of elements of X can be defined for every $q > 0$ in the same way as before. Suppose that

$$\rho(x, x') \le d(x, x') \tag{16.8}$$

for every $x, x' \in X$. If $z_1, \ldots, z_n$ is any finite sequence of elements of X, then

$$(\rho, q)\text{-length of } z_1, \ldots, z_n \le (d, q)\text{-length of } z_1, \ldots, z_n \tag{16.9}$$

for every $q > 0$. Let $\rho_q(\cdot, \cdot)$ be the q-semimetrification of $\rho(\cdot, \cdot)$ on X, as in (16.4). Observe that

$$\rho_q(z_1, z_n) \le (\rho, q)\text{-length of } z_1, \ldots, z_n \le (d, q)\text{-length of } z_1, \ldots, z_n \tag{16.10}$$

for every finite sequence $z_1, \ldots, z_n$ of elements of X, using the definition of $\rho_q(\cdot, \cdot)$ in the first step, and (16.9) in the second step. It follows that

$$\rho_q(x, x') \le d_q(x, x') \tag{16.11}$$

for every $x, x' \in X$, by the definition (16.4) of $d_q(x, x')$. In particular, if $\rho(\cdot, \cdot)$ is a q-semimetric on X for some $q > 0$, then $\rho_q = \rho$, as in (16.7). In this case, we get that

$$\rho(x, x') \le d_q(x, x') \tag{16.12}$$

for every $x, x' \in X$, so that $d_q(\cdot, \cdot)$ is the largest q-semimetric on X that is less than or equal to $d(\cdot, \cdot)$.

As a basic example, suppose that X is the real line, and that

$$d(x, x') = |x - x'| \tag{16.13}$$

is the standard metric on $\mathbf{R}$, associated to the standard absolute value function $|\cdot|$. If $q > 1$, then it is easy to see that

$$d_q(x, x') = 0 \tag{16.14}$$

for every $x, x' \in \mathbf{R}$, using finite sequences $z_1, \ldots, z_n$ such that $z_{j+1} - z_j$ is small for each j.

17 Lipschitz conditions

Let X, Y be sets, and let d_X, d_Y be nonnegative real-valued functions on $X \times X$, $Y \times Y$, respectively, that vanish on the corresponding diagonals and are symmetric, as in (1.1) and (1.2). Also let f be a mapping from X into Y, and observe that

$$d_Y(f(x), f(x')) \tag{17.1}$$

is another nonnegative real-valued function on $X \times X$ that vanishes on the diagonal and is symmetric, as in (1.1) and (1.2). If d_Y is a q-semimetric on Y for some $q > 0$, then it is easy to see that (17.1) is a q-semimetric on X.

Let α be a positive real number, and suppose that f is Lipschitz of order α with respect to d_X, d_Y. As in Section 6, this means that there is a nonnegative real number C such that

$$d_Y(f(x), f(x')) \le C\, d_X(x, x')^\alpha \tag{17.2}$$

for every $x, x' \in X$. Of course, the present discussion is a bit different from the previous one, since we are not asking that d_X, d_Y be q_X, q_Y-semimetrics for some $q_X, q_Y > 0$. In the following, it will sometimes be convenient to restrict our attention to $C > 0$, although we can still take C to be arbitrarily small. Note that (17.2) holds with $C = 0$ if and only if

$$d_Y(f(x), f(x')) = 0 \tag{17.3}$$

for every $x, x' \in X$, which does not involve α or d_X.

Let $q > 0$ be given, and let $d_{Y,q}$ be the q-semimetrification of d_Y on Y, as in the previous section. In particular, $d_{Y,q}$ is a q-semimetric on Y, and

$$0 \le d_{Y,q}(y, y') \le d_Y(y, y') \tag{17.4}$$

for every $y, y' \in Y$, as in (16.5). Combining this with (17.2), we get that

$$d_{Y,q}(f(x), f(x')) \le C\, d_X(x, x')^\alpha \tag{17.5}$$

for every $x, x' \in X$. If $C = 0$, then this says that

$$d_{Y,q}(f(x), f(x')) = 0 \tag{17.6}$$

for every $x, x' \in X$, which could also be derived from (17.3) and (17.4). Otherwise, if $C > 0$, then (17.5) can be reformulated as saying that

$$C^{-1/\alpha}\, d_{Y,q}(f(x), f(x'))^{1/\alpha} \le d_X(x, x') \tag{17.7}$$

for every $x, x' \in X$.

It is easy to see that

$$d_{Y,q}(y, y')^{1/\alpha} \tag{17.8}$$

is a $(q\,\alpha)$-semimetric on Y, because $d_{Y,q}$ is a q-semimetric on Y. Hence

$$d_{Y,q}(f(x), f(x'))^{1/\alpha} \tag{17.9}$$

defines a $(q\,\alpha)$-semimetric on X, as in (17.1). If $C > 0$, then $C^{-1/\alpha}$ times (17.9) is a $(q\,\alpha)$-semimetric on X as well. Let $d_{X,q\,\alpha}$ be the $(q\,\alpha)$-semimetrification of d_X on X, as in the preceding section again. If $C > 0$, then (17.7) implies that

$$C^{-1/\alpha}\, d_{Y,q}(f(x), f(x'))^{1/\alpha} \le d_{X,q\,\alpha}(x, x') \tag{17.10}$$

for every $x, x' \in X$, as in (16.12). Equivalently, this means that

$$d_{Y,q}(f(x), f(x')) \le C\, d_{X,q\,\alpha}(x, x')^\alpha \tag{17.11}$$

for every $x, x' \in X$ under these conditions. Note that (17.11) still works when $C = 0$, either by applying this argument to C small and positive, or using (17.6).

18 Repeated semimetrification

Let X be a set, and let d_X be a nonnegative real-valued function on $X \times X$ that satisfies (1.1) and (1.2). Also let $d_{X,q}$ be the q-semimetrification of d_X on X for each $q > 0$, as in (16.4). In particular, $d_{X,q}$ satisfies (1.1) and (1.2) too, and so we can repeat the process. Thus, for each $q_1, q_2 > 0$, we let d_{X,q_1,q_2} be the q_2-semimetrification of d_{X,q_1}, as in (16.4) again.

Of course, d_{X,q_1} is a q_1-semimetric on X, by construction, as in Section 16. If $0 < q \le q_1$, then it follows that d_{X,q_1} is a q-semimetric on X as well, as in Section 1. This implies that

$$d_{X,q_1,q_2} = d_{X,q_1} \tag{18.1}$$

when $0 < q_2 \le q_1$, because the q-semimetrification of a q-semimetric on X is equal to itself, as in (16.7). Similarly, we would like to check that

$$d_{X,q_1,q_2} = d_{X,q_2} \tag{18.2}$$

when $0 < q_1 \le q_2$. Let us begin by observing that

$$d_{X,q_1,q_2} \le d_{X,q_1} \le d_X \tag{18.3}$$

for any $q_1, q_2 > 0$, where both steps correspond to (16.5). This implies that

$$d_{X,q_1,q_2} \le d_{X,q_2}, \tag{18.4}$$

as in (16.12), because d_{X,q_1,q_2} is a q_2-semimetric on X. In the other direction, we have that

$$d_{X,q_2} \le d_X, \tag{18.5}$$

using (16.5) again. Remember that d_{X,q_2} is a q_2-semimetric on X, and hence a q-semimetric on X for every $q \le q_2$, as in Section 1. In particular, d_{X,q_2} is a q_1-semimetric on X when $q_1 \le q_2$. Thus (18.5) implies that

$$d_{X,q_2} \le d_{X,q_1} \tag{18.6}$$

in this situation, as in (16.12). It follows that

$$d_{X,q_2} \le d_{X,q_1,q_2}, \tag{18.7}$$

as in (16.12) again, because d_{X,q_2} is a q_2-semimetric on X. This shows that (18.2) holds when $q_1 \le q_2$, by combining (18.4) and (18.7).

Let Y be another set, and let d_Y be a nonnegative real-valued function on $Y \times Y$ that satisfies (1.1) and (1.2). As before, we let $d_{Y,q}$ be the q-semimetrification of d_Y on Y for each $q > 0$, as in (16.4). Suppose that f is a mapping from X into Y that satisfies

$$d_Y(f(x), f(x')) \le C\, d_X(x, x')^\alpha \tag{18.8}$$

for some $\alpha > 0$ and $C \ge 0$, and for every $x, x' \in X$, as in (17.2). This implies that (17.11) holds for every $q > 0$, as in the previous section. Let $q > 0$ be given, and let $C(q)$ be the smallest nonnegative real number such that

$$d_{Y,q}(f(x), f(x')) \le C(q)\, d_{X,q\,\alpha}(x, x')^\alpha \tag{18.9}$$

for every $x, x' \in X$. More precisely, one can take $C(q)$ to be the infimum of the set of nonnegative real numbers for which such an estimate holds. This set contains C, because of (17.11). It follows that the infimum exists, and satisfies

$$C(q) \le C \tag{18.10}$$

for every $q > 0$.

We would like to check that

$$C(q_2) \le C(q_1) \tag{18.11}$$

when $0 < q_1 \le q_2$, so that $C(q)$ is monotonically decreasing in q. By definition of $C(q_1)$, we have that

$$d_{Y,q_1}(f(x), f(x')) \le C(q_1)\, d_{X,q_1\,\alpha}(x, x')^\alpha \tag{18.12}$$

for every $x, x' \in X$, which is the same as (18.9) with $q = q_1$. In order to get (18.11), it suffices to verify that

$$d_{Y,q_2}(f(x), f(x')) \le C(q_1)\, d_{X,q_2\,\alpha}(x, x')^\alpha \tag{18.13}$$

for every $x, x' \in X$ when $q_1 \le q_2$, since $C(q_2)$ is supposed to be the smallest nonnegative real number for which such an estimate holds. Let d_{Y,q_1,q_2} be the q_2-semimetrification of d_{Y,q_1} on Y, as before. Using (18.12), one can check that

$$d_{Y,q_1,q_2}(f(x), f(x')) \le C(q_1)\, d_{X,q_1\,\alpha,q_2\,\alpha}(x, x')^\alpha \tag{18.14}$$

for every $x, x' \in X$. More precisely, this follows from (17.11), with d_{Y,q_1} in place of d_Y, $d_{X,q_1\,\alpha}$ in place of d_X, $C(q_1)$ in place of C, q_2 in place of q, and (18.12) in place of (17.2). If $q_1 \le q_2$, then we also have that

$$d_{Y,q_1,q_2} = d_{Y,q_2} \quad \text{and} \quad d_{X,q_1\,\alpha,q_2\,\alpha} = d_{X,q_2\,\alpha}, \tag{18.15}$$

as in (18.2). Thus (18.14) is the same as (18.13), as desired.

19 q-Seminormification

Let k be a field, and let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$. Also let V be a vector space over k, and let N be a nonnegative real-valued function on V that satisfies the homogeneity condition (5.1) with respect to $|\cdot|$ on k. If $v_1, \ldots, v_l$ is a finite sequence of vectors in V, then define their (N,q)-*sum* to be

$$\Big(\sum_{j=1}^{l} N(v_j)^q\Big)^{1/q} \tag{19.1}$$

when $0 < q < \infty$, and

$$\max_{1 \le j \le l} N(v_j) \tag{19.2}$$

when $q = \infty$. Note that (19.2) is less than or equal to (19.1) for each q, and that (19.1) is monotonically decreasing in q, as in (11.9). If N is a q-seminorm on V for some $q > 0$, then

$$N\Big(\sum_{j=1}^{l} v_j\Big) \le (N,q)\text{-sum of } v_1, \ldots, v_l \tag{19.3}$$

for every finite sequence $v_1, \ldots, v_l$ of vectors in V, by the q-seminorm version of the triangle inequality.

If $v \in V$ and $0 < q \le \infty$, then put

$$N_q(v) = \inf\Big\{(N,q)\text{-sums of finite sequences } v_1, \ldots, v_l \text{ of vectors in } V \text{ such that } v = \sum_{j=1}^{l} v_l\Big\}. \tag{19.4}$$

Observe that

$$0 \le N_q(v) \le N(v) \tag{19.5}$$

for every $v \in V$ and $q > 0$, since we can take $l = 1$ and $v_1 = v$ in the infimum on the right side of (19.4). We also have that

$$N_q(t\, v) = |t|\, N_q(v) \tag{19.6}$$

for every $v \in V$, $t \in k$, and $q > 0$, because of the analogous hypothesis on N. It is easy to see that N_q satisfies the q-seminorm version of the triangle inequality for every $q > 0$, by construction, so that N_q is a q-seminorm on V for each $q > 0$. The monotonicity of (N,q)-sums in q mentioned in the previous paragraph implies that $N_q(v)$ is monotonically decreasing in q for each $v \in V$ as well. If N is a q-seminorm on V for some $q > 0$, then

$$N(v) \le N_q(v) \tag{19.7}$$

for every $v \in V$, by (19.3). This implies that

$$N_q(v) = N(v) \tag{19.8}$$

for every $v \in V$ in this case, using also (19.5).

Let $\widetilde{N}$ be another nonnegative real-valued function on V that satisfies the homogeneity condition

$$\widetilde{N}(t\, v) = |t|\, \widetilde{N}(v) \tag{19.9}$$

for every $v \in V$ and $t \in k$, and suppose that

$$\widetilde{N}(v) \le N(v) \tag{19.10}$$

for every $v \in V$. If $v_1, \ldots, v_l$ is a finite sequence of vectors in V, then

$$(\widetilde{N}, q)\text{-sum of } v_1, \ldots, v_l \le (N, q)\text{-sum of } v_1, \ldots, v_l \tag{19.11}$$

for every $q > 0$, where $(\widetilde{N}, q)$-sums are defined in the same way as before, using $\widetilde{N}$. This implies that

$$\widetilde{N}_q(v) \le N_q(v) \tag{19.12}$$

for every $v \in V$ and $q > 0$, where $\widetilde{N}_q$ is defined as in (19.4). If $\widetilde{N}$ is a q-seminorm on V for some $q > 0$, so that $\widetilde{N}_q = \widetilde{N}$, as in (19.8), then we get that

$$\widetilde{N}(v) \le N_q(v) \tag{19.13}$$

for every $v \in V$. Thus N_q is the largest q-seminorm on V that is less than or equal to N.

The homogeneity condition (5.1) implies in particular that $N(0) = 0$ and $N(-v) = N(v)$ for each $v \in V$, since $|0| = 0$ and $|-1| = 1$, as in (4.1) and (4.6). Hence

$$d(v, v') = N(v - v') \tag{19.14}$$

defines a nonnegative real-valued function on $V \times V$ that vanishes on the diagonal and is symmetric in v and v', as in (1.1) and (1.2). Let $z_1, \ldots, z_l$ be a finite sequence of elements of V, whose (d, q)-length can be defined for every $q > 0$ as in Section 16. If we put

$$v_j = z_{j+1} - z_j \tag{19.15}$$

for each $j = 1, \ldots, l - 1$, then

$$d(z_j, z_{j+1}) = N(z_{j+1} - z_j) = N(v_j) \tag{19.16}$$

for every $j = 1, \ldots, l - 1$, and

$$z_l - z_1 = \sum_{j=1}^{l-1} v_j, \tag{19.17}$$

where the sum on the right side of (19.17) is interpreted as being equal to 0 when $l = 1$. It follows that

$$(19.18) \qquad (d,q)\text{-length of } z_1, \ldots, z_l = (N,q)\text{-sum of } v_1, \ldots, v_{l-1}$$

for every $q > 0$, where both sides of (19.18) are interpreted as being equal to 0 when $l = 1$. Similarly, if $v_1, \ldots, v_{l-1}$ is any finite sequence of elements of V, then put $z_1 = 0$ and

$$(19.19) \qquad z_r = \sum_{j=1}^{r-1} v_j$$

for $r = 2, \ldots, l$. This defines a finite sequence $z_1, \ldots, z_l$ of elements of V that satisfies (19.15) for each $j = 1, \ldots, l-1$, so that (19.18) holds for every $q > 0$. It follows that

$$(19.20) \qquad d_q(v, v') = N_q(v - v')$$

for every $v, v' \in V$ and $q > 0$, where $d_q(v, v')$ is as in (16.4).

20 q-Seminormification, continued

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$ again, and let V, W be vector spaces over k. Also let N_V, N_W be nonnegative real-valued functions on V, W, respectively, that satisfy the homogeneity condition (5.1) with respect to $|\cdot|$ on k. If T is a linear mapping from V into W, then

$$(20.1) \qquad N_W(T(v))$$

is another nonnegative real-valued function on V that satisfies the homogeneity condition (5.1). If N_W is a q-seminorm on W for some $q > 0$, then (20.1) is a q-seminorm on V too.

Suppose that T is bounded with respect to N_V and N_W, in the sense that there is a nonnegative real number C such that

$$(20.2) \qquad N_W(T(v)) \le C\, N_V(v)$$

for every $v \in V$. This is the same as in Section 13, except that N_V, N_W are not required to be q-seminorms for any $q > 0$. Let $q > 0$ be given, and let $N_{W,q}$ be the q-seminormification of N_W on W, as in the previous section. Thus $N_{W,q}$ is a q-seminorm on W which is less than or equal to N_W, so that

$$(20.3) \qquad N_{W,q}(T(v)) \le C\, N_V(v)$$

for every $v \in V$, by (20.2). Let us now use this to verify that

$$(20.4) \qquad N_{W,q}(T(v)) \le C\, N_{V,q}(v)$$

for every $v \in V$, where $N_{V,q}$ is the q-seminormification of N_V on V, as before.

If $C = 0$, then (20.3) implies (20.4) trivially, and so we may as well restrict our attention to $C > 0$. In this case, (20.3) is the same as saying that

$$C^{-1} N_{W,q}(T(v)) \le N_V(v) \tag{20.5}$$

for every $v \in V$. Note that

$$C^{-1} N_{W,q}(T(v)) \tag{20.6}$$

is a q-seminorm on V, because $N_{W,q}$ is a q-seminorm on W. Using this and (20.6), we get that

$$C^{-1} N_{W,q}(T(v)) \le N_{V,q}(v) \tag{20.7}$$

for every $v \in V$, as in (19.13). More precisely, here we take N to be N_V in (19.13), and we take $\widetilde{N}$ to be (20.6). Of course, (20.7) implies (20.4), as desired. This argument is analogous to the one in Section 17, and one could reduce the previous version using (19.20).

Let X be a nonempty set, and let $\ell^r(X,k)$ be the space of r-summable k-valued functions on X for each $r > 0$, as in Section 10. More precisely, we use the given q_k-absolute value function $|\cdot|$ on k as a q_k-norm on k, where k is considered as a one-dimensional vector space over itself. Remember that

$$\ell^r(X,k) \subseteq \ell^q(X,k) \tag{20.8}$$

when $r \le q$, as in (11.7), in which case we have that

$$\|f\|_q \le \|f\|_r \tag{20.9}$$

for every $f \in \ell^r(X,k)$, as in (11.9). If $r \le q_k$, then $\|f\|_r$ is an r-norm on $\ell^r(X,k)$, as in Section 10, and similarly $\|f\|_q$ is a q-norm on $\ell^q(X,k)$ when $q \le q_k$. If $r \le q \le q_k$, then it follows that $\|f\|_q$ is less than or equal to the q-seminormification of $\|f\|_r$ on $\ell^r(X,k)$, because of (20.9). One can also check directly that the q-seminormification of $\|f\|_r$ on $\ell^r(X,k)$ is less than or equal to $\|f\|_q$ when $r \le q$. This is simpler when f has finite support in X, and otherwise one can basically approximate $f \in \ell^r(X,k)$ by functions with finite support on X. Thus the q-seminormification of $\|f\|_r$ on $\ell^r(X,k)$ is equal to $\|f\|_q$ when $r \le q \le q_k$.

21 Repeated seminormification

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a vector space over k. Also let N_V be a nonnegative real-valued function on V that satisfies the homogeneity condition (5.1) with respect to $|\cdot|$ on k, and let $N_{V,q}$ be the q-seminormification of N_V on V for each $q > 0$, as in (19.4). As before, we can repeat the process, to define the q_2-seminormification N_{V,q_1,q_2} of N_{V,q_1} for every $q_1, q_2 \ge 0$. We would like to check that

$$N_{V,q_1,q_2} = N_{V,q_1} \tag{21.1}$$

when $q_2 \le q_1$, and

(21.2) $$N_{V,q_1,q_2} = N_{V,q_2}$$

when $q_1 \le q_2$. Of course, this is analogous to the earlier discussion for semimetrics, and one could reduce to that situation using (19.20).

Remember that N_{V,q_1} is a q_1-seminorm on V, by construction, as in Section 19. This implies that N_{V,q_1} is a q-seminorm on V when $0 < q \le q_1$, as in Section 5. If $q_2 \le q_1$, then it follows that (21.1) holds, because the q-seminormification of a q-seminorm is itself, as in (19.8). In order to deal with the case where $q_1 \le q_2$, observe first that

(21.3) $$N_{V,q_1,q_2} \le N_{V,q_1} \le N_V$$

for every $q_1, q_2 > 0$, where both steps correspond to (19.5). This implies that

(21.4) $$N_{V,q_1,q_2} \le N_{V,q_2},$$

as in (19.13), since N_{V,q_1,q_2} is a q_2-seminorm on V. To get the opposite inequality, observe that

(21.5) $$N_{V,q_2} \le N_V,$$

using (19.5) again. If $q_1 \le q_2$, then N_{V,q_2} is a q_1-seminorm on V, as in Section 5, and because N_{V,q_2} is a q_2-seminorm on V. Hence (21.5) implies that

(21.6) $$N_{V,q_2} \le N_{V,q_1},$$

as in (19.13). Using (19.13) again, we get that

(21.7) $$N_{V,q_2} \le N_{V,q_1,q_2},$$

since N_{V,q_2} is a q_2-seminorm on V. Combining (21.4) and (21.7), we get that (21.2) holds when $q_1 \le q_2$, as desired.

Let W be another vector space over k, and let N_W be a nonnegative real-valued function on W that satisfies the usual homogeneity condition (5.1) with respect to $|\cdot|$ on k. Also let $N_{W,q}$ be the q-seminormification of N_W on W for each $q > 0$, as in (19.4), and let N_{W,q_1,q_2} be the q_2-semimetrification of N_{W,q_1} for any $q_1, q_2 > 0$. Of course, N_{W,q_1,q_2} can be characterized as in (21.1) and (21.2), as before. Consider the space

(21.8) $$\mathcal{BL}_q(V, W)$$

of bounded linear mappings from V into W with respect to $N_{V,q}$ on V and $N_{W,q}$ on W, for each $q > 0$. This is a vector space over k, as in Section 13, and we let

(21.9) $$\|T\|_{op,q} = \|T\|_{op,VW,q}$$

be the corresponding operator q-seminorm of such a linear mapping T.

If a linear mapping T from V into W satisfies (20.2) for some $C \ge 0$ and every $v \in V$, then T is an element of $\mathcal{BL}_q(V, W)$ for every $q > 0$, and

(21.10) $$\|T\|_{op,q} \le C,$$

by (20.4). Similarly, suppose that $T \in \mathcal{BL}_{q_1}(V, W)$ for some $q_1 > 0$, so that

$$N_{W,q_1}(T(v)) \le \|T\|_{op,q_1}\, N_{V,q_1}(v) \tag{21.11}$$

for every $v \in V$. If $q_1 \le q_2$, then it follows that

$$N_{W,q_2}(T(v)) \le \|T\|_{op,q_1}\, N_{V,q_2}(v) \tag{21.12}$$

for every $v \in V$, as in (20.4) again. More precisely, (21.11) plays the role of (20.2) here, and we are also using (21.2) and its analogue for W. This shows that

$$\mathcal{BL}_{q_1}(V, W) \subseteq \mathcal{BL}_{q_2}(V, W), \tag{21.13}$$

when $q_1 \le q_2$, and that

$$\|T\|_{op,q_2} \le \|T\|_{op,q_1}. \tag{21.14}$$

for every $T \in \mathcal{BL}_{q_1}(V, W)$.

22 Some simple quotient spaces

Let X be a set, and let $d_X(\cdot, \cdot)$ be a q_X-semimetric on X for some $q_X > 0$. If $x, x' \in X$ satisfy

$$d_X(x, x') = 0, \tag{22.1}$$

then put

$$x \sim_X x'. \tag{22.2}$$

This defines a binary relation on X, which is in fact an equivalence relation on X. Let

$$\widehat{X} = X/\sim_X \tag{22.3}$$

be the corresponding quotient space, which is the set of equivalence classes in X determined by (22.2). If $x \in X$, then let $[x]_X$ be the equivalence class in X that contains x. Of course, if $d_X(\cdot, \cdot)$ is a q-metric on X, then (22.1) holds if and only if $x = x'$. In this case, $[x] = \{x\}$ for every $x \in X$, and $\widehat{X}$ is essentially the same as X.

If $x, x', w, w' \in X$ satisfy $x \sim_X x'$ and $w \sim_X w'$, then it is easy to see that

$$d_X(x, w) = d_X(x', w'), \tag{22.4}$$

using the q-semimetric version of the triangle inequality. This implies that there is a nonnegative real-valued function $d_{\widehat{X}}(\cdot, \cdot)$ defined on $\widehat{X} \times \widehat{X}$ such that

$$d_{\widehat{X}}([x]_X, [w]_X) = d_X(x, w) \tag{22.5}$$

for every $x, w \in X$. One can also check that $d_{\widehat{X}}(\cdot, \cdot)$ is a q_X-semimetric on $\widehat{X}$, because of the corresponding properties of $d_X(\cdot, \cdot)$ on X. If $x, w \in X$ and $[x] \ne [w]$, then (22.5) is strictly positive, by construction. This means that $d_{\widehat{X}}(\cdot, \cdot)$ is a q-metric on $\widehat{X}$.

Let Y be another set with a q_Y-semimetric $d_Y(\cdot,\cdot)$ for some $q_Y > 0$. Thus we can define an equivalence relation $\sim_Y$ on Y as before, and we let $\widehat{Y} = Y/\sim_Y$ be the corresponding quotient space of equivalence classes in Y. If $y \in Y$, then we let $[y]_Y$ be the equivalence class in Y determined by $\sim_Y$ that contains y, and we let $d_{\widehat{Y}}(\cdot,\cdot)$ be the q_Y-metric induced on $\widehat{Y}$ by $d_Y(\cdot,\cdot)$ on Y as in the preceding paragraph. Let f be a mapping from X into Y, and let $\widetilde{f}$ be the composition of f with the natural quotient mapping from Y onto $\widehat{Y}$. Thus $\widetilde{f}$ is the mapping from X into $\widehat{Y}$ defined by putting

$$\widetilde{f}(x) = [f(x)]_Y \tag{22.6}$$

for each $x \in X$. Suppose that for every $x, x' \in X$ with $x \sim_X x'$, we have that

$$f(x) \sim_Y f(x'), \tag{22.7}$$

which is the same as saying that

$$f([x]_X) \subseteq [f(x)]_Y \tag{22.8}$$

for every $x \in X$. Equivalently, this means that $\widetilde{f}$ is constant on the equivalence classes in X determined by $\sim_X$. This implies that there is a mapping $\widehat{f}$ from $\widehat{X}$ into $\widehat{Y}$ such that

$$\widehat{f}([x]_X) = \widetilde{f}(x) \tag{22.9}$$

for every $x \in X$.

Let f be a mapping from X into Y again. Suppose that f is continuous at a point $x \in X$, with respect to the topologies determined on X and Y by d_X and d_Y, respectively. This implies that (22.7) holds for every $x' \in X$ with $x \sim_X x'$, which is the same as saying that (22.8) holds. If f is a continuous mapping from X into Y, then this holds for every $x \in X$. This implies that there is a mapping $\widehat{f}$ from $\widehat{X}$ into $\widehat{Y}$ as in (22.9), as before. In this case, one can check that $\widehat{f}$ is also continuous as a mapping from $\widehat{X}$ into $\widehat{Y}$, with respect to the topologies determined on $\widehat{X}$ and $\widehat{Y}$ by $d_{\widehat{X}}$ and $d_{\widehat{Y}}$, respectively. Similarly, if f is uniformly continuous with respect to d_X and d_Y, then $\widehat{f}$ is uniformly continuous with respect to $d_{\widehat{X}}$ and $d_{\widehat{Y}}$. If f satisfies a Lipschitz condition with respect to d_X and d_Y, then $\widehat{f}$ satisfies the same type of condition with respect to $d_{\widehat{X}}$ and $d_{\widehat{Y}}$.

23 Simple quotient spaces, continued

Let k be a field with a q_k-absolute value function for some $q_k > 0$, and let V be a vector space over k with a q_V-seminorm N_V with respect to $|\cdot|$ on k for some $q_V > 0$. Under these conditions, it is easy to see that

$$V_0 = \{v \in V : N_V(v) = 0\} \tag{23.1}$$

is a linear subspace of V. This leads to an equivalence relation on V, where $v, v' \in V$ are equivalent with respect to this relation if and only if

$$v - v' \in V_0. \tag{23.2}$$

If $v \in V$, then the equivalence class in V that contains v is

$$v + V_0 = \{v + u : u \in V_0\}. \tag{23.3}$$

Let V/V_0 be the corresponding quotient space of equivalence classes in V. It is well known that V/V_0 is also a vector space over k in a natural way, so that the quotient mapping

$$v \mapsto v + V_0 \tag{23.4}$$

is linear as a mapping from V onto V/V_0. Remember that N_V determines a q-semimetric on V, as in (5.6). In this situation, the equivalence relation on V determined by (23.2) is the same as the one determined by the q-semimetric associated to N_V, as in the previous section. If N_V is a q-norm on V, then $V_0 = \{0\}$, and V/V_0 is essentially the same as V.

If $v, v' \in V$ satisfy (23.2), then one can check that

$$N_V(v) = N_V(v'), \tag{23.5}$$

using the q_V-semimetric version of the triangle inequality. It follows that there is a nonnegative real-valued function N_{V/V_0} on V/V_0 such that

$$N_{V/V_0}(v + V_0) = N_V(v) \tag{23.6}$$

for every $v \in V$. One can also verify that N_{V/V_0} is a q_V-seminorm on V/V_0, using the corresponding properties of N_V on V. More precisely, N_{V/V_0} is a q_V-norm on V/V_0, by construction. The q-metric on V/V_0 associated to N_{V/V_0} corresponds exactly to the quotient q-metric determined by the q-semimetric associated to N_V on V as in the previous section.

Let W be another vector space over k, and let N_W be a q_W-seminorm on W with respect to $|\cdot|$ on k for some $q_W > 0$. This leads to a linear subspace

$$W_0 = \{w \in W : N_W(w) = 0\} \tag{23.7}$$

of W as before, and the corresponding quotient vector space W/W_0 over k. If $w \in W$, then $w + W_0$ is the equivalence class in W corresponding to the equivalence relation associated to W_0 that contains w, which is an element of W/W_0. Let N_{W/W_0} be the q_W-norm on W/W_0 determined by N_W on W as in the preceding paragraph, so that

$$N_{W/W_0}(w + W_0) = N_W(w) \tag{23.8}$$

for every $w \in W$. If T is a linear mapping from V into W that satisfies

$$T(V_0) \subseteq W_0, \tag{23.9}$$

then there is an induced linear mapping $\widehat{T}$ from V/V_0 into W/W_0 such that

$$\widehat{T}(v + V_0) = T(v) + W_0 \tag{23.10}$$

for every $v \in V$.

Suppose that T is a bounded linear mapping from V into W, with respect to N_V and N_W, respectively, so that

$$N_W(T(v)) \le C\, N_V(v) \tag{23.11}$$

for some $C \ge 0$ and every $v \in V$. This implies that (23.9) holds, and hence that there is an induced linear mapping $\widehat{T}$ from V/V_0 into W/W_0 as in (23.10). Note that

$$N_{W/W_0}(\widehat{T}(v + V_0)) = N_{W/W_0}(T(v) + W_0) = N_W(T(v)) \tag{23.12}$$

for every $v \in V$, because of (23.8) and (23.10). It follows that

$$N_{W/W_0}(\widehat{T}(v + V_0)) \le C\, N_{V/V_0}(v + V_0) \tag{23.13}$$

for every $v \in V$, using the same constant C as in (23.11). Conversely, if (23.13) holds for some $C \ge 0$, then it is easy to see that (23.11) holds with the same C. More precisely, this works for any linear mapping T from V into W that satisfies (23.9), so that $\widehat{T}$ can be defined as in (23.10). Under these conditions, we get that T is a bounded linear mapping from V into W with respect to N_V and N_W if and only if $\widehat{T}$ is bounded as a linear mapping from V/V_0 into W/W_0 with respect to N_{V/V_0} and N_{W/W_0}, in which case the operator norms of T and $\widehat{T}$ are the same.

Part II
Invertible linear mappings

24 Preliminary remarks

Let k be a field, and let V, W be vector spaces over k. If T is a one-to-one linear mapping from V onto W, then the inverse T^{-1} of T is also linear as a mapping from W into V. Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let N_V, N_W be q_V, q_W-seminorms on V, W, respectively, for some $q_V, q_W > 0$, and with respect to $|\cdot|$ on k. Also let T be a bounded linear mapping from V into W with respect to N_V and N_W, as in Section 13. If T is a one-to-one mapping from V onto W, and if the inverse mapping T^{-1} is bounded as a linear mapping from W into V, then T is considered to be invertible as a bounded linear mapping from V onto W.

Let T be a linear mapping from V into W again, and suppose that there is a nonnegative real number C such that

$$N_V(v) \le C\, N_W(T(v)) \tag{24.1}$$

for every $v \in V$. If N_V is a q_V-norm on V, then (24.1) implies that T is one-to-one on V. If T is one-to-one on V, then the inverse mapping T^{-1} can be defined as a linear mapping from $T(V) \subseteq W$ onto V. In this case, (24.1) says exactly that T^{-1} is bounded as a linear mapping from $T(V)$ onto V, using the restriction of N_W to $T(V)$ as the domain of T^{-1}. Of course, if V and W are finite-dimensional vector spaces over k with the same dimension, and if T is a one-to-one linear mapping from V into W, then T maps V onto W.

Let V_0, W_0 be the linear subspaces of V and W defined in (23.1), (23.7). If T is a bounded linear mapping from V into W, then T satisfies (23.9), and T induces a linear mapping $\widehat{T}$ from V/V_0 into W/W_0 as in (23.10), as before. Similarly, if T is a one-to-one mapping from V onto W, and if the inverse mapping T^{-1} is bounded from W into V, then T^{-1} maps W_0 into V_0, and induces a linear mapping $\widehat{(T^{-1})}$ from W/W_0 into V/V_0. Under these conditions, we get that

$$T(V_0) = W_0, \tag{24.2}$$

and that $\widehat{T}$ is a one-to-one mapping from V/V_0 onto W/W_0. More precisely, (24.2) implies that

$$T^{-1}(W_0) = V_0 \tag{24.3}$$

here, because T is injective, and (24.3) implies that $\widehat{T}$ is injective too. The inverse of $\widehat{T}$ as a mapping from V/V_0 onto W/W_0 is the same as the mapping $\widehat{(T^{-1})}$ from W/W_0 onto V/V_0 induced by T^{-1}, which is to say that

$$\widehat{T}^{-1} = \widehat{(T^{-1})}. \tag{24.4}$$

The boundedness of T and T^{-1} between V and W imply the boundedness of $\widehat{T}$ and $\widehat{(T^{-1})}$ between V/V_0 and W/W_0, as in the previous section. Thus we get that $\widehat{T}$ is a bounded linear mapping from V/V_0 onto W/W_0 with bounded inverse in this situation, because of (24.4).

Let T be a linear mapping from V into W that satisfies (24.1) for some $C \geq 0$ again. In particular, this means that

$$N_W(T(v)) = 0 \quad \text{implies} \quad N_V(v) = 0 \tag{24.5}$$

for every $v \in V$, so that

$$T^{-1}(W_0) \subseteq V_0. \tag{24.6}$$

Suppose that $T(V_0) \subseteq W_0$, so that T induces a linear mapping $\widehat{T}$ from V/V_0 into W/W_0, as before. In this case, (24.1) implies that

$$N_{V/V_0}(v + V_0) \leq C\, N_{W/W_0}(\widehat{T}(v + V_0)) \tag{24.7}$$

for every $v \in V$, by (23.6) and (23.12). Conversely, (24.7) implies (24.1) when T satisfies (23.9), for essentially the same reasons.

25 Density and surjectivity

Let X be a set with a q-metric $d(\cdot, \cdot)$ for some $q > 0$, and let Y be a subset of X. Thus Y may also be considered as a q-metric space, using the restriction of $d(\cdot, \cdot)$ to elements of Y. If X is complete as a q-metric space, and if Y is a closed set in X with respect to the topology determined by $d(\cdot, \cdot)$, then it is easy to see that Y is complete as a q-metric space too. Conversely, if Y is complete as a q-metric space, then Y has to be a closed set in X. To see this, let $\{y_j\}_{j=1}^\infty$ be a sequence of elements of Y that converges to an element x of X. This implies that $\{y_j\}_{j=1}^\infty$ is a Cauchy sequence in X, and hence in Y as well. If Y is complete, then $\{y_j\}_{j=1}^\infty$ converges to an element y of Y. The limit of a convergent sequence in a q-metric space is unique, as in the case of ordinary metric spaces, so that $x = y \in Y$, as desired.

Now let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be vector spaces over k with q_V, q_W-norms N_V, N_W, respectively, for some $q_V, q_W > 0$, and with respect to $|\cdot|$ on k. Also let T be a bounded linear mapping from V into W that satisfies (24.1) for some $C \geq 0$ and every $v \in V$. We have already seen that (24.1) implies that T is injective on V in this situation, and that T^{-1} is bounded on $T(V)$, as in the previous section. If $T(V) = W$, then it follows that T is invertible as a bounded linear mapping from V into W. In particular, this holds when V and W are finite-dimensional vector spaces over k, with the same dimension.

If T is a bounded linear mapping from V into W that satisfies (24.1), and if V is complete with respect to the q_V-metric associated to N_V, then it is easy to see that $T(V)$ is complete with respect to the q_W-metric associated to the restriction of N_W to $T(V)$. More precisely, any sequence of elements of $T(V)$ can be expressed as $\{T(v_j)\}_{j=1}^\infty$ for some sequence $\{v_j\}_{j=1}^\infty$ of elements of V. If $\{T(v_j)\}_{j=1}^\infty$ is a Cauchy sequence in $T(V)$, then (24.1) implies that $\{v_j\}_{j=1}^\infty$ is a Cauchy sequence in V. If V is complete, then it follows that $\{v_j\}_{j=1}^\infty$ converges to some $v \in V$. If T is bounded, then $\{T(v_j)\}_{j=1}^\infty$ converges to $T(v)$ in $T(V)$, as desired. This implies that $T(V)$ is a closed set in W with respect to the q_W-metric associated to N_W, by the remarks at the beginning of the section. If $T(V)$ is also dense in W with respect to the q_W-metric associated to N_W, then it follows that $T(V) = W$ under these conditions.

Let us now give a criterion for a linear subspace W_1 of W to be dense in W. In this discussion, we can let N_W be a q_W-seminorm on W, instead of a q_W-norm. Let a be a real number with $0 \leq a < 1$. Suppose that for each $w \in W$ there is a $u_1 \in W_1$ such that

$$N_W(w - u_1) \leq a\, N_W(w). \tag{25.1}$$

We can also apply this condition to $w - u_1$ instead of w, to get that there is a $u_2 \in W_1$ such that

$$N_W(w - u_1 - u_2) \leq a\, N_W(w - u_1) \leq a^2\, N_W(w). \tag{25.2}$$

Continuing in this way, it follows that for each positive integer l, there are elements $u_1, \ldots, u_l$ of W_1 such that

$$N_W\Big(w - \sum_{j=1}^{l} u_j\Big) \le a^l \, N_W(w). \tag{25.3}$$

This implies that W_1 is dense in W with respect to the q_W-semimetric associated to N_W, since $\sum_{j=1}^{l} u_j \in W_1$ for each l, and $a^l \to 0$ as $l \to \infty$.

26 Small perturbations

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$ again, and let V, W be vector spaces over k with q_V, q_W-seminorms N_V, N_W, respectively, for some $q_V, q_W > 0$, and with respect to $|\cdot|$ on k. Also let T be a linear mapping from V into W that satisfies (24.1) for some $C \ge 0$, and let R be a bounded linear mapping from V into W. We would like to verify that $R + T$ satisfies a condition like (24.1) when

$$C \, \|R\|_{op,VW} < 1, \tag{26.1}$$

where $\|R\|_{op,VW}$ is the operator q_W-seminorm of R corresponding to N_V on V and N_W on W, as in Section 13.

If $q_W < \infty$, then

$$\begin{aligned} N_V(v)^{q_W} &\le C^{q_W} \, N_W(T(v))^{q_W} \\ &\le C^{q_W} \, (N_W(R(v) + T(v))^{q_W} + N_W(R(v))^{q_W}) \\ &\le C^{q_W} \, N_W((R+T)(v))^{q_W} + C^{q_W} \, \|R\|_{op,VW}^{q_W} \, N_V(v)^{q_W} \end{aligned} \tag{26.2}$$

for every $v \in V$. This uses (24.1) in the first step, the q_W-seminorm version of the triangle inequality for N_W in the second step, and the boundedness of R in the third step. It follows that

$$(1 - C^{q_W} \, \|R\|_{op,VW}^{q_W}) \, N_V(v)^{q_W} \le C^{q_W} \, N_W((R+T)(v))^{q_W} \tag{26.3}$$

for every $v \in V$. If (26.1) holds, then we get that

$$N_V(v) \le C \, (1 - C^{q_w} \, \|R\|_{op,VW}^{q_W})^{-1/q_W} \, N_W((R+T)(v)) \tag{26.4}$$

for every $v \in V$.

Similarly, if $q_W = \infty$, then

$$\begin{aligned} N_V(v) &\le C \, N_W(T(v)) \\ &\le C \, \max(N_W(R(v) + T(v)), N_W(R(v))) \\ &\le \max(C \, N_W((R+T)(v)), C \, \|R\|_{op,VW} \, N_V(v)) \end{aligned} \tag{26.5}$$

for every $v \in V$. This uses (24.1) in the first step, the ultrametric version of the triangle inequality in the second step, and the boundedness of R in the third step. If (26.1) holds, then (26.5) implies that

$$N_V(v) \le C \, N_W((R+T)(v)) \tag{26.6}$$

for every $v \in V$. More precisely, (26.6) is trivial when $N_V(v) = 0$, and so it suffices to check (26.6) when $N_V(v) > 0$. In this case, (26.1) implies that

$$C \, \|R\|_{op,VW} \, N_V(v) < N_V(v), \tag{26.7}$$

so that (26.6) follows from (26.5).

Observe that

$$N_W(R(v)) \le \|R\|_{op,VW} \, N_V(v) \le C \, \|R\|_{op,VW} \, N_W(T(v)) \tag{26.8}$$

for every $v \in V$, using the boundedness of R in the first step, and (24.1) in the second step. This implies that

$$N_W(T(v) - (R+T)(v)) = N_W(R(v)) \le C \, \|R\|_{op,VW} \, N_W(T(v)) \tag{26.9}$$

for every $v \in V$. Put

$$W_1 = (R+T)(V) \tag{26.10}$$

and

$$a = C \, \|R\|_{op,VW}, \tag{26.11}$$

so that (26.1) says exactly that $a < 1$. If $T(V) = W$, then (26.9) implies that (25.1) holds, and it follows that (26.10) is dense in W with respect to the q_W-semimetric associated to N_W, as in the previous section. Similarly, if $T(V)$ is dense in W, and (26.1) holds, then one can check that (26.10) is dense in W too.

27 Small perturbations, continued

Let k be a field, and let V be a vector space over k. If T is a linear mapping from V into itself, then we let T^j be the jth power of T on V with respect to composition for each positive integer j, and we interpret this as being the identity operator I on V when $j = 0$. It is well known and easy to see that

$$(I - T) \sum_{j=0}^{n} T^j = \Big(\sum_{j=0}^{n} T^j \Big) (I - T) = I - T^{n+1} \tag{27.1}$$

for each nonnegative integer n, using composition of mappings as multiplication. If $I - T$ is invertible on V, then it follows that

$$\sum_{j=0}^{n} T^j = (I - T)^{-1} \, (I - T^{n+1}) = (I - T^{n+1}) \, (I - T)^{-1} \tag{27.2}$$

for each nonnegative integer n. This implies that

$$(I - T)^{-1} - \sum_{j=0}^{n} T^j = T^{n+1} \, (I - T)^{-1} = (I - T)^{-1} \, T^{n+1} \tag{27.3}$$

for each $n \geq 0$.

Let $|\cdot|$ be a q-absolute value function on k for some $q > 0$, and let N be a q-seminorm on V with respect to $|\cdot|$ on k. Remember that $\mathcal{BL}(V)$ denotes the corresponding vector space of bounded linear mappings on V, as in Section 13, equipped with the associated operator q-seminorm $\|\cdot\|_{op}$. If T is a bounded linear mapping on V, then

$$\|T^j\|_{op} \leq \|T\|_{op}^j \tag{27.4}$$

for every positive integer j, by (13.6). This holds when $j = 0$ too, as in (13.7), with the right side of (27.4) interpreted as being equal to 1. In particular, if

$$\|T\|_{op} < 1, \tag{27.5}$$

then it follows that

$$\lim_{j \to \infty} \|T^j\|_{op} = 0. \tag{27.6}$$

If $q < \infty$, then

$$\left\| \sum_{j=0}^{n} T^j \right\|_{op}^q \leq \sum_{j=0}^{n} \|T^j\|_{op}^q \leq \sum_{j=0}^{n} \|T\|_{op}^{q\,j} \tag{27.7}$$

for each $n \geq 0$, using (27.4) in the second step. Of course,

$$\sum_{j=0}^{\infty} \|T\|_{op}^{q\,j} = (1 - \|T\|_{op}^q)^{-1} \tag{27.8}$$

when T satisfies (27.5), by the usual formula for the sum of a geometric series. Thus

$$\left\| \sum_{j=0}^{n} T^j \right\|_{op} \leq (1 - \|T\|_{op}^q)^{-1/q} \tag{27.9}$$

for every $n \geq 0$ when $q < \infty$ and (27.5) holds. If $q = \infty$, then

$$\left\| \sum_{j=0}^{n} T^j \right\|_{op} \leq \max_{0 \leq j \leq n} \|T^j\|_{op} \leq \max_{0 \leq j \leq n} \|T\|_{op}^j \tag{27.10}$$

for each $n \geq 0$, using (27.4) in the second step again. This implies that

$$\left\| \sum_{j=0}^{n} T^j \right\|_{op} \leq 1 \tag{27.11}$$

for every $n \geq 0$ when $q = \infty$ and $\|T\|_{op} \leq 1$.

Suppose for the moment that $I - T$ is invertible on V, with a bounded inverse. If T satisfies (27.5) and hence (27.6), then

$$\lim_{n \to \infty} \left\| (I - T)^{-1} - \sum_{j=0}^{n} T^j \right\|_{op} = 0, \tag{27.12}$$

by (27.3). It follows that

$$\|(I - T)^{-1}\|_{op} \le (1 - \|T\|_{op}^q)^{-1/q} \tag{27.13}$$

when $q < \infty$ and (27.5) holds, because of (27.9). If $q = \infty$, then we get that

$$\|(I - T)^{-1}\|_{op} \le 1 \tag{27.14}$$

when (27.5) holds, using (27.11). These estimates for the operator q-seminorm of $(I - T)^{-1}$ could also be obtained from the discussion in Sections 24 and 26 under these conditions.

28 Convergence of the series

Let k be a field with a q-absolute value function $|\cdot|$ for some $q > 0$, and let V be a vector space over k with a q-norm N with respect to $|\cdot|$ on k. Thus $\|\cdot\|_{op}$ defines a q-norm on $\mathcal{BL}(V)$ too, as in Section 13. Let T be a bounded linear mapping from V into itself that satisfies

$$\|T\|_{op} < 1 \tag{28.1}$$

again. If $I - T$ has a bounded inverse on V, then

$$(I - T)^{-1} = \sum_{j=0}^{\infty} T^j, \tag{28.2}$$

by (27.12). More precisely, (27.12) says exactly that the sequence of partial sums $\sum_{j=0}^{n} T^j$ converges to $(I - T)^{-1}$ with respect to the q-metric on $\mathcal{BL}(V)$ associated to the operator q-norm. In the context of the previous section, this was only a q-semimetric on $\mathcal{BL}(V)$, and so the limit was not necessarily unique. Here the limit is unique, because $\|\cdot\|_{op}$ is a q-norm on $\mathcal{BL}(V)$, and the convergence of the infinite series is defined in terms of the convergence of the sequence of partial sums, as in Section 8.

Suppose now that T is a bounded linear operator on V such that

$$\sum_{j=0}^{\infty} T^j \tag{28.3}$$

converges in $\mathcal{BL}(V)$ with respect to the operator q-norm $\|\cdot\|_{op}$, as in Section 8. In particular, this implies that

$$\lim_{j \to \infty} \|T^j\|_{op} = 0, \tag{28.4}$$

as in (8.6). This permits us to take the limit as $n \to \infty$ in (27.1), to get that

$$(I - T) \sum_{j=0}^{\infty} T^j = \Big(\sum_{j=0}^{\infty} T^j\Big) (I - T) = I. \tag{28.5}$$

This also uses the hypothesis that N and hence $\|\cdot\|_{op}$ be q-norms, and not just q-seminorms, in order to have uniqueness of limits. It follows that $I - T$ has a bounded inverse on V under these conditions, with the inverse given as in (28.2).

Suppose that V is complete with respect to the q-metric associated to N, so that $\mathcal{BL}(V)$ is complete with respect to the q-metric associated to $\|\cdot\|_{op}$, as in Section 13. If T is a bounded linear operator on V that satisfies (28.1) and $q < \infty$, then we have that

$$\sum_{j=0}^{\infty} \|T^j\|_{op}^q \le \sum_{j=0}^{\infty} \|T\|_{op}^{q\,j} = (1 - \|T\|_{op}^q)^{-1}, \tag{28.6}$$

as in (27.4) and (27.8). This means that (28.3) converges q-absolutely with respect to the operator q-norm $\|\cdot\|_{op}$, and hence that (28.3) converges in $\mathcal{BL}(V)$ with respect to the operator q-norm, because of completeness, as in Section 8. Of course, if T satisfies (28.1), then (28.4) holds, because of (27.4). If $q = \infty$, then (28.4) implies that (28.3) converges in $\mathcal{BL}(V)$ with respect to the operator ultranorm $\|\cdot\|_{op}$, because of completeness, as in Section 8 again. In both cases, the convergence of (28.3) implies that $I - T$ has a bounded inverse on V, as before. The invertibility of $I-T$ on V when T satisfies (28.1) and V is complete could also be derived from the discussion in Sections 24, 25, and 26.

If (28.3) converges in $\mathcal{BL}(V)$ with respect to the operator q-norm $\|\cdot\|_{op}$, then we have that

$$(I - T)^{-1} - I = \sum_{j=1}^{\infty} T^j, \tag{28.7}$$

by subtracting the $j = 0$ term from both sides of (28.2). If $q < \infty$ and T satisfies (28.1), then it follows that

$$\|(I - T)^{-1} - I\|_{op}^q \le \sum_{j=1}^{\infty} \|T^j\|_{op}^q \le \sum_{j=1}^{\infty} \|T\|_{op}^{q\,j} = \|T\|_{op}^q\,(1 - \|T\|_{op}^q)^{-1}, \tag{28.8}$$

using (27.4) in the second step. Similarly, if $q = \infty$ and T satisfies (28.1), then we get that

$$\|(I - T)^{-1} - I\|_{op} \le \max_{j \ge 1} \|T^j\|_{op} \le \max_{j \ge 1} \|T\|_{op}^j \le \|T\|_{op}, \tag{28.9}$$

using (27.4) in the second step again. These estimates could also be obtained in the context of the previous section, assuming that $I - T$ has a bounded inverse on V instead of the convergence of (28.3), and using (27.12). Another version of this will be considered in Section 31.

29 Some variants

Let k be a field, let V, W be vector spaces over k. If R is a linear mapping from V into W and T is a linear mapping from W into V, then the composition

$T \circ R$ is defined as a linear mapping from V into itself, and $R \circ T$ is defined as a linear mapping from W into itself. As usual, we say that $T_1 : W \to V$ is a left-inverse of R if

$$T_1 \circ R = I_V, \tag{29.1}$$

and that $T_2 : W \to V$ is a right-inverse of R if

$$R \circ T_2 = I_W, \tag{29.2}$$

where I_V, I_W are the identity mappings on V and W, respectively. If R has both a left-inverse T_1 and a right-inverse T_2, then

$$T_1 = T_1 \circ (R \circ T_2) = (T_1 \circ R) \circ T_2 = T_2, \tag{29.3}$$

which implies that R is invertible as a mapping from V into W. Note that the left-invertibility of R implies that R is injective, and the right-invertibility of R implies that R is surjective. If V, W are finite-dimensional vector spaces over k with the same dimension, then a linear mapping R from V into W is injective if and only if it is surjective. In this case, it follows that R is invertible when R is left-invertible or right-invertible.

Let $R : V \to W$ and $T : W \to V$ be linear mappings, and suppose that $T \circ R$ is invertible as a linear mapping from V into itself. This implies that

$$((T \circ R)^{-1} \circ T) \circ R = (T \circ R)^{-1} \circ (T \circ R) = I_V, \tag{29.4}$$

and

$$T \circ (R \circ (T \circ R)^{-1}) = (T \circ R) \circ (T \circ R)^{-1} = I_V \tag{29.5}$$

so that R is left-invertible and T is right-invertible. Similarly, if $R \circ T$ is invertible on W, then

$$R \circ (T \circ (R \circ T)^{-1}) = (R \circ T) \circ (R \circ T)^{-1} = I_W \tag{29.6}$$

and

$$((R \circ T)^{-1} \circ R) \circ T = (R \circ T)^{-1} \circ (R \circ T) = I_W, \tag{29.7}$$

which implies that R is right-invertible and T is left-invertible. If $R \circ T$ and $T \circ R$ are both invertible, then it follows that R and T are both invertible. If V and W are finite-dimensional vector spaces over k with the same dimension, and if $R \circ T$ or $T \circ R$ are invertible, then R and T are invertible, because they each have a one-sided inverse, as before.

Let us now take $V = W$, and let R and T be linear mappings from V into itself. If there is a one-sided inverse to R on V that commutes with R, then R is invertible on V. If R is invertible on V, then it is easy to see that R^{-1} commutes with any linear mapping on V that commutes with R. If R and T are commuting linear mappings on V, then the invertibility of $R \circ T$ and $T \circ R$ are the same. In this case, $R \circ T$ commutes with R and T, so that $(R \circ T)^{-1}$ commutes with R and T as well when $R \circ T$ is invertible on V.

Let T be a linear mapping from V into itself again, and suppose that

$$I - T^{n+1} \tag{29.8}$$

is invertible on V for some nonnegative integer n. Of course, powers of T commute with each other, which implies that (29.8) commutes with powers of T. It follows that the inverse of (29.8) commutes with powers of T too, as well as with finite sums of powers of T. Using (27.1), we get that $I - T$ is also invertible on V under these conditions, with

$$(I - T)^{-1} = \Big(\sum_{j=0}^{n} T^j\Big)\,(I - T^{n+1})^{-1} = (I - T^{n+1})^{-1} \sum_{j=0}^{n} T^j. \tag{29.9}$$

Let $|\cdot|$ be a q-absolute value function on k for some $q > 0$, let N be a q-seminorm on V with respect to $|\cdot|$ on k, and let $\|\cdot\|_{op}$ be the corresponding operator q-seminorm on the space $\mathcal{BL}(V)$ of bounded linear mappings on V with respect to N, as in Section 13. If T is a bounded linear mapping on V with respect to N, and if (29.8) has a bounded inverse on V for some $n \geq 0$, then (29.9) implies that $I - T$ has a bounded inverse on V too. More precisely, we get that

$$\|(I - T)^{-1}\|_{op} \leq \|(I - T^{n+1})^{-1}\|_{op} \, \Big\|\sum_{j=0}^{n} T^j\Big\|_{op} \tag{29.10}$$

under these conditions. Remember that the second factor on the right side of (29.10) can be estimated as in (27.7) when $q < \infty$, and as in (27.10) when $q = \infty$.

30 Some variants, continued

Let k be a field with a q-absolute value function $|\cdot|$ for some $q > 0$ again, and let V be a vector space over k with a q-seminorm N with respect to $|\cdot|$ on k. This leads to the corresponding operator q-seminorm $\|\cdot\|_{op}$ on the space $\mathcal{BL}(V)$ of bounded linear mappings from V into itself with respect to N, as in Section 13. Let T be a bounded linear mapping on V with respect to N, and suppose that

$$\lim_{j \to \infty} \|T^j\|_{op} = 0. \tag{30.1}$$

In particular, this implies that $\|T^j\|_{op} < 1$ for all but finitely many positive integers j. Conversely, suppose that

$$\|T^{j_0}\|_{op} < 1 \tag{30.2}$$

for some $j_0 \in \mathbf{Z}_+$. Observe that

$$\|T^{j_0\, l + r}\|_{op} \leq \|T^{j_0}\|_{op}^{l} \, \|T\|_{op}^{r} \tag{30.3}$$

for all nonnegative integers l, r, with the usual conventions when $l = 0$ or $r = 0$. This implies that (30.1) holds, since it suffices to consider $r < j_0$.

Suppose now that N is a q-norm on V, so that $\|\cdot\|_{op}$ is a q-norm on $\mathcal{BL}(V)$. Suppose also that V is complete with respect to the q-metric associated to N, so that $\mathcal{BL}(V)$ is complete with respect to the q-metric associated to $\|\cdot\|_{op}$, as in Section 13. Let T be a bounded linear mapping on V that satisfies (30.1) again. If $q = \infty$, then it follows that (28.3) converges in $\mathcal{BL}(V)$ with respect to $\|\cdot\|_{op}$, as in Section 8. This implies that $I - T$ has a bounded inverse on V, with the inverse given in (28.2), as in Section 28.

If $q < \infty$, then let j_0 be a positive integer such that (30.2) holds, as before. Note that every nonnegative integer can be expressed in a unique way as

$$j = j_0\, l + r, \tag{30.4}$$

where l, r are nonnegative integers, and $r < j_0$. This implies that

$$\sum_{j=0}^{\infty} \|T^j\|_{op}^q = \sum_{r=0}^{j_0-1} \sum_{l=0}^{\infty} \|T^{j_0\, l+r}\|_{op}^q. \tag{30.5}$$

Using (30.3), we get that this is less than or equal to

$$\Big(\sum_{r=0}^{j_0-1} \|T\|_{op}^{q\, r}\Big) \Big(\sum_{l=0}^{\infty} \|T^{j_0}\|_{op}^{q\, l}\Big). \tag{30.6}$$

The sum over r is obviously finite, and the sum over l is a convergent geometric series, because of (30.2). Thus

$$\sum_{j=0}^{\infty} \|T^j\|_{op}^q < \infty, \tag{30.7}$$

which means that (28.3) converges q-absolutely with respect to $\|\cdot\|_{op}$. It follows that (28.3) converges in $\mathcal{BL}(V)$ with respect to $\|\cdot\|_{op}$ under these conditions, by completeness, as in Section 8. This implies that $(I - T)$ has a bounded inverse on V in this case too, as in Section 28.

Alternatively, we can use (30.2) and completeness to get that

$$I - T^{j_0} \tag{30.8}$$

has a bounded inverse on V, as in Section 28. This implies that $I - T$ has a bounded inverse on V, as in the previous section.

31 Small perturbations, revisited

Let k be a field, and let V, W be vector spaces over k. Also let T_1, T_2 be one-to-one linear mappings from V onto W. Thus the corresponding inverse mappings T_1^{-1}, T_2^{-1} are defined as linear mappings from W onto V. Observe that

$$T_1^{-1} - T_2^{-1} = T_1^{-1} \, T_2 \, T_2^{-1} - T_1^{-1} \, T_1 \, T_2^{-1} = T_1^{-1} \, (T_2 - T_1) \, T_2^{-1}, \tag{31.1}$$

as linear mappings from W into V. Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let N_V, N_W be a q_V, q_W-seminorms on V, W, respectively, for some $q_V, q_W > 0$, and with respect to $|\cdot|$ on k. This leads to corresponding spaces $\mathcal{BL}(V, W)$ and $\mathcal{BL}(W, V)$ of bounded linear mappings from V into W and from W into V, as in Section 13, with their associated operator q_W, q_V-seminorms $\|\cdot\|_{op,VW}$ and $\|\cdot\|_{op,WV}$, respectively. Suppose that T_1, T_2 are bounded linear mappings from V onto W with bounded inverses. In this case, (31.1) implies that

$$\|T_1^{-1} - T_2^{-1}\|_{op,WV} \le \|T_1^{-1}\|_{op,WV} \, \|T_2 - T_1\|_{op,VW} \, \|T_2^{-1}\|_{op,WV}. \tag{31.2}$$

Suppose for the moment that

$$\|T_1^{-1}\|_{op,WV} \, \|T_1 - T_2\|_{op,VW} < 1. \tag{31.3}$$

We would like to use the discussion in Section 26 with $T = T_1$ and $R = T_2 - T_1$, so that $R + T = T_2$. In this situation, (24.1) holds with $C = \|T_1^{-1}\|_{op,WV}$, and hence (31.3) corresponds to (26.1). If $q_W < \infty$, then we have that

$$\|T_2^{-1}\|_{op,WV} \le \|T_1^{-1}\|_{op,WV} \, (1 - \|T_1^{-1}\|_{op,WV}^{q_W} \, \|T_1 - T_2\|_{op,VW}^{q_W})^{-1/q_W}, \tag{31.4}$$

as in (26.4). Combining this with (31.2), we get that

$$\begin{aligned} &\|T_1^{-1} - T_2^{-1}\|_{op,WV} \\ &\quad \le \|T_1^{-1}\|_{op,WV}^2 \, (1 - \|T_1^{-1}\|_{op,WV}^{q_W} \, \|T_1 - T_2\|_{op,VW}^{q_W})^{-1/q_W} \, \|T_1 - T_2\|_{op,VW} \end{aligned} \tag{31.5}$$

when (31.3) holds. Similarly, if $q_W = \infty$ and (31.3) holds, then

$$\|T_2^{-1}\|_{op,WV} \le \|T_1^{-1}\|_{op,WV}, \tag{31.6}$$

as in (26.6). Combining this with (31.2) again, we get that

$$\|T_1^{-1} - T_2^{-1}\|_{op,WV} \le \|T_1^{-1}\|_{op,WV}^2 \, \|T_1 - T_2\|_{op,VW} \tag{31.7}$$

in this case.

Suppose now that N_V is a q_V-norm on V, so that $\|\cdot\|_{op,WV}$ is a q_V-norm on $\mathcal{BL}(W, V)$. Let T_1 be a bounded linear mapping from V onto W with bounded inverse again. Note that this implies that N_W is a q_W-norm on W, and hence that $\|\cdot\|_{VW}$ is a q_W-norm on $\mathcal{BL}(V, W)$. Also let T_2 be a bounded linear mapping from V into W that satisfies (31.3). Under these conditions, T_2 is injective, the inverse of T_2 is bounded on $T_2(V)$, and $T_2(V)$ is dense in W with respect to the q_W-metric associated to N_W, as in Sections 24 and 26. If V has finite dimension, then W has finite dimension equal to the dimension of V, because of the invertibility of T_1. In this case, the injectivity of T_2 implies that T_2 is surjective. Similarly, if V is complete, then the surjectivity of T_2 follows from the discussion in Section 25 in this situation.

Let us now consider the case where $V = W$, with $N_V = N_W$ and $q_V = q_W$. Of course, the identity mapping $I = I_V$ on V is a bounded linear mapping with bounded inverse, and so the previous remarks can be applied to $T_1 = I$. This is closely connected to the discussion in Sections 27 and 28. Note that one can reduce to this case, by composing T_2 with T_1^{-1}.

32 Isometric linear mappings

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be vector spaces over k with q_V, q_W-seminorms N_V, N_W, respectively, for some $q_V, q_W > 0$, and with respect to $|\cdot|$ on k. Also let $\|\cdot\|_{op,VW}$ and $\|\cdot\|_{op,WV}$ be the corresponding operator q_W and q_V-seminorms on $\mathcal{BL}(V,W)$ and $\mathcal{BL}(W,V)$, respectively, as in Section 13. A linear mapping T from V into W is said to be an *isometry* with respect to N_V and N_W if

$$N_W(T(v)) = N_V(v) \tag{32.1}$$

for every $v \in V$. Of course, this is the same as saying that

$$N_W(T(v)) \le N_V(v) \tag{32.2}$$

and

$$N_V(v) \le N_W(T(v)) \tag{32.3}$$

for every $v \in V$. The first condition (32.2) means exactly that T is a bounded linear mapping from V into W, with

$$\|T\|_{op,VW} \le 1. \tag{32.4}$$

If T is injective, then the second condition (32.3) means that T^{-1} is a bounded linear mapping from $T(V)$ onto V, using the restriction of N_W to $T(V)$, and that the corresponding operator q_V-seminorm of T^{-1} is less than or equal to 1. In particular, if T is a one-to-one linear mapping from V onto W, then (32.4) is the same as saying that T^{-1} is bounded, with

$$\|T^{-1}\|_{op,WV} \le 1. \tag{32.5}$$

Note that (32.3) implies that T is injective when N_V is a q_V-norm on V.

Suppose that T_1, T_2 are one-to-one linear mappings from V onto W. If T_1, T_2 are isometries too, then (31.2) implies that

$$\|T_1^{-1} - T_2^{-1}\|_{op,WV} \le \|T_1 - T_2\|_{op,VW}, \tag{32.6}$$

since the operator q_V-seminorms of T_1^{-1}, T_2^{-1} are less than or equal to 1, as in (32.5). In fact, we have that

$$\|T_1^{-1} - T_2^{-1}\|_{op,WV} = \|T_1 - T_2\|_{op,VW} \tag{32.7}$$

under these conditions. This is because the reverse inequality in (32.6) can be obtained by applying the analogous inequality to T_1^{-1}, T_2^{-1}. One can also derive (32.7) more directly from (31.1) in this situation.

Suppose now that $q_W = \infty$, so that N_W is a semi-ultranorm on W, and $\|\cdot\|_{op,VW}$ is a semi-ultranorm on $\mathcal{BL}(V, W)$. Let T be an isometric linear mapping from V into W, and let R be a bounded linear mapping from V into W such that

$$\|R\|_{op,VW} < 1. \tag{32.8}$$

This implies that $R + T$ is a bounded linear mapping from V into W as well, with

$$\|R + T\|_{op,VW} \le \max(\|R\|_{op,VW}, \|T\|_{op,VW}) \le 1. \tag{32.9}$$

We also have that

$$N_V(v) \le N_W((R + T)(v)) \tag{32.10}$$

for every $v \in V$ in this situation, as in (26.6). More precisely, T satisfies (24.1) with $C = 1$, so that (26.1) reduces to (32.8). This permits us to get (32.10), as in Section 26. It follows that $R + T$ is also an isometric linear mapping from V into W under these conditions.

Let us now take $V = W$, $N_V = N_W$, and $q_V = q_W = \infty$. Let R_1, R_2 be bounded linear mappings on V, and observe that

$$(I + R_1) \circ (I + R_2) = I + R_1 + R_2 + R_1 \circ R_2. \tag{32.11}$$

If $\|\cdot\|_{op}$ denotes the corresponding operator semi-ultranorm on $\mathcal{BL}(V)$, then we get that

$$\begin{aligned} \|R_1 + R_2 + R_1 \circ R_2\|_{op} &\le \max(\|R_1\|_{op}, \|R_2\|_{op}, \|R_1 \circ R_2\|_{op}) \\ &\le \max(\|R_1\|_{op}, \|R_2\|_{op}, \|R_1\|_{op}\,\|R_2\|_{op}). \end{aligned} \tag{32.12}$$

If we also ask that

$$\min(\|R_1\|_{op}, \|R_2\|_{op}) \le 1, \tag{32.13}$$

then it follows that

$$\|R_1 + R_2 + R_1 \circ R_2\|_{op} \le \max(\|R_1\|_{op}, \|R_2\|_{op}). \tag{32.14}$$

In particular, if $\|R_1\|_{op}, \|R_2\|_{op} < 1$, then the left side of (32.14) is strictly less than 1 too.

33 Groups of linear mappings

Let k be a field, and let V be a vector space over k. It is well known that the collection $\mathcal{GL}(V)$ of one-to-one linear mappings from V onto itself is a group, with composition of mappings as multiplication, and the identity operator I on V as the identity element. Let $|\cdot|$ be a q-absolute value function on k for some $q > 0$, and let N be a q-seminorm on V with respect to $|\cdot|$ on k. The collection $\mathcal{BGL}(V)$ of one-to-one bounded linear mappings from V onto itself with bounded inverse with respect to N on V is also a group, which may be considered as a subgroup of $\mathcal{GL}(V)$.

Let $\|\cdot\|_{op}$ be the operator q-seminorm on the space $\mathcal{BL}(V)$ of bounded linear mappings on V associated to N as in Section 13. This leads to a q-semimetric on $\mathcal{BL}(V)$ in the usual way, and thus a topology on $\mathcal{BL}(V)$. Using this topology, one can define the corresponding product topology on

$$\mathcal{BL}(V) \times \mathcal{BL}(V). \tag{33.1}$$

Note that composition of linear mappings on V defines a mapping from (33.1) into $\mathcal{BL}(V)$, which is being used here as multiplication on $\mathcal{BL}(V)$. One can check that this mapping is continuous with respect to the product topology on (33.1), by standard arguments.

Similarly,

$$T \mapsto T^{-1} \tag{33.2}$$

defines a mapping from $\mathcal{GL}(V)$ into itself, which sends $\mathcal{BGL}(V)$ to itself. Let us take $\mathcal{BGL}(V)$ to be equipped with the topology induced by the topology on $\mathcal{BL}(V)$ corresponding to $\|\cdot\|_{op}$ as in the previous paragraph. We have already seen that (33.2) is continuous as a mapping on $\mathcal{BGL}(V)$ with respect to this topology in Section 31. More precisely, the continuity of this mapping at a point $T_1 \in \mathcal{BGL}(V)$ follows from (31.5) when $q_W < \infty$, and from (31.7) when $q_W = \infty$. In both cases, we ask that $T_2 \in \mathcal{BGL}(V)$ be sufficiently close to T_1, as in (31.3).

Let us also take

$$\mathcal{BL}(V) \cap \mathcal{GL}(V) \tag{33.3}$$

to be equipped with the topology induced by the one on $\mathcal{BL}(V)$ corresponding to $\|\cdot\|_{op}$ as before. The discussion in Sections 24 and 26 implies that $\mathcal{BGL}(V)$ is a relatively open set in (33.3). Suppose for the moment that N is a q-norm on V, so that $\|\cdot\|_{op}$ is a q-norm on $\mathcal{BL}(V)$. If V has finite dimension over k, or if V is complete with respect to the q-metric associated to N, then $\mathcal{BGL}(V)$ is already an open set in $\mathcal{BL}(V)$. This was basically mentioned in Section 31, in slightly different terms.

34 Subgroups of linear isometries

Let k be a field with a q-absolute value function for some $q > 0$, and let V be a vector space over k with a q-seminorm N with respect to $|\cdot|$ on k. The collection of one-to-one isometric linear mappings from V onto itself is a subgroup of the group $\mathcal{BGL}(V)$ of bounded linear mappings on V with bounded inverses on V defined in the previous section. Of course, if N is a q-norm on V, then every isometric linear mapping from V into itself is injective, as in Section 32. If V has finite dimension over k, then every injective linear mapping from V into itself is a surjection. Thus every isometric linear mapping from V into itself is a one-to-one mapping from V onto itself when N is a q-norm on V and V has finite dimension over k.

Suppose from now on in this section that N is a semi-ultranorm on V. In this case, we have seen in Section 32 that the collection of isometric linear mappings from V into itself is an open set in $\mathcal{BL}(V)$ with respect to the topology corresponding to the operator q-seminorm $\|\cdot\|_{op}$ associated to N. This implies that the collection of one-to-one isometric linear mappings from V onto itself is relatively open in (33.3). Suppose for the moment that N is an ultranorm on V, so that isometric linear mappings from V into itself are injective. If V has finite dimension over k, then isometric linear mappings from V into itself are bijective, and the collection of these mappings is an open set in $\mathcal{BL}(V)$. Otherwise, if V is complete with respect to the ultranorm associated to N, then $\mathcal{BGL}(V)$ is an open set in $\mathcal{BL}(V)$, as in the previous section. In this situation, we get that the collection of isometric linear mappings from V onto itself is an open set in $\mathcal{BL}(V)$.

Consider the set

$$\{T \in \mathcal{BL}(V) : \|T - I\|_{op} < 1\}. \tag{34.1}$$

Because N is a semi-ultranorm on V, every element of (34.1) is an isometry on V, as in Section 32. We have also seen in Section 32 that the composition of two elements of (34.1) is an element of (34.1) as well. If a one-to-one linear mapping T from V onto itself is an element of (34.1), then T^{-1} is an element of (34.1) too, by (32.7). It follows that

$$\{T \in \mathcal{BL}(V) \cap \mathcal{GL}(V) : \|T - I\|_{op} < 1\} \tag{34.2}$$

is a subgroup of the group of one-to-one isometric linear mappings from V onto itself. Suppose for the moment again that N is an ultranorm on V, which implies that the elements of (34.1) are injective. If V has finite dimension over k, then the elements of (34.1) are surjective as well, so that (34.1) is the same as (34.2). Similarly, if V is complete with respect to the ultrametric associated to N, then the elements of (34.1) are surjective, as in Section 31. This implies that (34.1) is the same as (34.2) in this case too.

Of course,

$$\{T \in \mathcal{BL}(V) : \|T - I\|_{op} < r\} \tag{34.3}$$

is contained in (34.1) when $0 < r \le 1$, and

$$\{T \in \mathcal{BL}(V) : \|T - I\|_{op} \le r\} \tag{34.4}$$

is contained in (34.1) when $0 < r < 1$. Thus the elements of (34.3) are isometries on V when $0 < r \le 1$, and the elements of (34.4) are isometries on V when $0 < r < 1$. It is easy to see that (34.3) and (34.4) are closed under compositions when $r \le 1$, using (32.14). If a one-to-one linear mapping T from V onto V is an element of (34.3) for some $r \le 1$, or an element of (34.4) for some $r < 1$, then T^{-1} has the same property, because of (32.7). This implies that the intersection of (34.3) with $\mathcal{GL}(V)$ is a subgroup of the group of one-to-one isometric linear mappings from V onto itself when $0 < r \le 1$, and that the intersection of (34.4) with $\mathcal{GL}(V)$ is a subgroup of the same group when $0 < r < 1$. If N is an ultranorm on V, and if V has finite dimension over k, then (34.3) is contained in $\mathcal{GL}(V)$ for each $r \le 1$, and (34.4) is contained in $\mathcal{GL}(V)$ for every $r < 1$, as in the preceding paragraph. The same conclusions also hold when N is an ultranorm on V and V is complete with respect to the associated ultrametric, as before.

35 Some additional remarks

Let k be a field with a q_k-absolute value function for some $q_k > 0$, and let V, W be vector spaces over k with q_V, q_W-seminorms N_V, N_W, respectively, for some $q_V, q_W > 0$, and with respect to $|\cdot|$ on k. This leads to the corresponding spaces $\mathcal{BL}(V,W)$ and $\mathcal{BL}(W,V)$ of bounded linear mappings from V into W and from W into V, respectively, with their associated operator q_W and q_V-seminorms $\|\cdot\|_{op,VW}$ and $\|\cdot\|_{op,WV}$, respectively, as in Section 13. Similarly, for each $q > 0$, let $N_{V,q}$ and $N_{W,q}$ be the q-seminormifications of N_V and N_W on V and W, respectively, as in (19.4). As before, this leads to the corresponding spaces

$$\mathcal{BL}_q(V,W) \quad \text{and} \quad \mathcal{BL}_q(W,V) \tag{35.1}$$

of bounded linear mappings from V into W and from W into V with respect to $N_{V,q}$ and $N_{W,q}$ for each $q > 0$, with their associated operator q-seminorms

$$\|\cdot\|_{op,VW,q} \quad \text{and} \quad \|\cdot\|_{op,WV,q}, \tag{35.2}$$

respectively. If $T \in \mathcal{BL}(V,W)$, then $T \in \mathcal{BL}_q(V,W)$ for every $q > 0$, and

$$\|T\|_{op,VW,q} \le \|T\|_{op,VW}. \tag{35.3}$$

This follows from the fact that (20.2) implies (20.4) for every $q > 0$. In particular,

$$\mathcal{BL}(V,W) \subseteq \mathcal{BL}_q(V,W) \tag{35.4}$$

for each $q > 0$. Of course, there are analogous statements with the roles of V and W exchanged.

Suppose now that T is a one-to-one linear mapping from V onto W. If

$$T \in \mathcal{BL}(V,W) \quad \text{and} \quad T^{-1} \in \mathcal{BL}(W,V), \tag{35.5}$$

then

$$T \in \mathcal{BL}_q(V,W) \quad \text{and} \quad T^{-1} \in \mathcal{BL}_q(W,V) \tag{35.6}$$

for every $q > 0$, as in the preceding paragraph. We also get that (35.3) holds for every $q > 0$, and similarly that

$$\|T^{-1}\|_{op,WV,q} \le \|T^{-1}\|_{op,WV} \tag{35.7}$$

for each $q > 0$. If T is also an isometry with respect to N_V and N_W on V and W, respectively, then

$$\|T\|_{op,VW},\ \|T^{-1}\|_{op,WV} \le 1, \tag{35.8}$$

as in (32.4) and (32.5). This implies that

$$\|T\|_{op,VW,q},\ \|T^{-1}\|_{op,WV,q} \le 1 \tag{35.9}$$

for every $q > 0$, by (35.3) and (35.7), so that T is an isometry with respect to $N_{V,q}$ on V and $N_{W,q}$ on W for each $q > 0$ too.

Let us now restrict our attention to the case where $V = W$, with $q_V = q_W$ and $N_V = N_W$. As usual, we let $\mathcal{BL}(V)$ be the space of bounded linear mappings from V into itself with respect to N_V, and we put

$$\mathcal{BL}_q(V) = \mathcal{BL}_q(V, V) \tag{35.10}$$

for each $q > 0$. Also let $\mathcal{BGL}(V)$ be the group of one-to-one linear mappings T from V onto itself such that T and T^{-1} are bounded with respect to N_V, as in Section 33, and let

$$\mathcal{BGL}_q(V) \tag{35.11}$$

be the analogous group of mappings with respect to $N_{V,q}$ for each $q > 0$. Thus

$$\mathcal{BGL}(V) \subseteq \mathcal{BGL}_q(V) \tag{35.12}$$

for every $q > 0$, since (35.5) implies (35.6). There is an analogous inclusion for the corresponding subgroups of isometries on V, by the remarks in the previous paragraph.

There are also appropriate monotonicity properties in q in this discussion, as in Section 21.

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