SEMINAR IN TOPOLOGY (18.904)

HOMEWORK 1 – SOLUTIONS

1. (a) Define the map
$$f\colon \mathbb{R}^2 \longrightarrow S^1 \times S^1$$
by
$$f(x, y) = (e^{2\pi i x}, e^{2\pi i y}).$$
It is obviously continuous, surjective, and makes the same identifications as $\sim$. It is also open (because it maps $(a, b) \times (c, d)$ into an open set in $S^1 \times S^1$), hence a quotient map. That means that the corresponding map
$$\mathbb{R}^2/{\sim} \longrightarrow S^1 \times S^1$$
is a homeomorphism.

   (b) The map
$$f\colon \mathbb{R}^2 \longrightarrow [0, \infty),$$
$$f(x, y) = \sqrt{x^2 + y^2},$$
is continuous, surjective, $f(x, y) = f(x', y') \iff (x, y) \sim (x', y')$. If V is a subset of $[0, \infty)$ with $f^{-1}(V) \subset \mathbb{R}^2$ open then $V = f^{-1}(V) \cap [0, \infty)$ is also open, so f is a quotient map, and it induces a homeomorphism between $\mathbb{R}^2/{\sim}$ and $[0, \infty)$.

2. Usually we take the torus to be $[0, 1] \times [0, 1]$ with some identification of the boundary, but the formulas are a bit simpler if we take the disk D^1 with some identification of the boundary. A deformation retraction $\big((D^1 \setminus \{0\})/{\sim}\big) \times I \to (D^1 \setminus \{0\})/{\sim}$ is for example the map induced by
$$f(x, y, t) = \lambda(x, y, t) \cdot (x, y)$$
with
$$\lambda(x, y, t) = 1 - t + \frac{t}{\sqrt{x^2 + y^2}}.$$
Obviously f is continuous, $f(x, y, 0) = (x, y)$, $f(x, y, 1) \in S^1$, $f(x, y, t) = (x, y)$ for $(x, y) \in S^1$.

3. If $r\colon X \to A$ is a retraction and $H\colon X \times I \to X$ a homotopy with $H(x, 0) = x$, $H(x, 1) = x_0$ for some $x_0 \in X$, then $G\colon A \times I \to A$, $G(a, t) = r(H(a, t))$, defines a homotopy between the identity and a constant map: G is obviously continuous, $G(a, 0) = r(H(a, 0)) = r(a) = a$, $G(a, 1) = r(H(a, 1)) = r(x_0)$. Therefore A is contractible.