SEMINAR IN TOPOLOGY (18.904)

HOMEWORK 2 – SOLUTIONS

1. Assume that β_h depends only on the endpoints of h, and pick $[f],[g] \in \pi_1(X)$. Then f is a loop, and hence $\beta_f = \beta_c$ where c is the constant map. But then $\beta_f([g]) = [fg\overline{f}] = [f][g][f]^{-1} = \beta_c([g]) = [g]$ and so $[f][g] = [g][f]$. Conversely, assume that $\pi_1(X)$ is abelian, and take paths g,h with $g(0) = h(0)$, $g(1) = h(1)$ and a loop f with $f(0) = f(1) = g(1) = h(1)$. By assumption, $[\overline{g}h]$ and $[f]$ commute, i.e. $[\overline{g}hf] = [f\overline{g}h]$, and hence $[hf\overline{h}] = [g\overline{g}hf\overline{h}] = [gf\overline{g}h\overline{h}] = [gf\overline{g}]$.

2. Let us prove $(a) \Rightarrow (b)$. Take a map $f\colon S^1 \to X$. By assumption, there is a homotopy $H\colon S^1 \times I \to X$ with $H(x,0) = x$ and $H(x,1) = x_0$. The map factors to a map $\tilde{f}\colon D := (S^1 \times I)/(S^1 \times \{1\}) \to X$, and it is obvious that

$$S^1 \times I \longrightarrow D^2,$$

$$(x,t) \mapsto (1-t)x,$$

induces a homeomorphism $D \approx D^2$ (the map is a quotient mapping because it is closed – it maps from a compact space into a Hausdorff space). Assume (b), and take a loop $f\colon I \to X$, $f(0) = f(1) = x_0$; then f induces a map $S^1 \to X$ (which we also denote by f), $f(1) = x_0$. There is an extension $\tilde{f}\colon D^2 \to X$, and obviously $H\colon I \times I \to X$, $H(t,s) = \tilde{f}((1-s)e^{2\pi it})$ is a homotopy between f and a constant map. This implies $\pi_1(X,x_0) = 0$. To complete the proof, assume that (c) holds. Every map $f\colon S^1 \to X$ represents a loop, and is therefore homotopic to a constant map.
If X is simply connected, then $\pi_1(X,x_0) = 0$ for any x_0, and hence (by $(c) \Rightarrow (a)$) every map $S^1 \to X$ are homotopic to a constant map. Because X is also path-connected, all constant maps are homotopic.
Conversely, if all maps $S^1 \to X$ are homotopic, they are homotopic to a constant map, and so (by $(a) \Rightarrow (c)$) $\pi_1(X,x_0) = 0$ for any x_0. Also, since all constant maps are homotopic, X must be path-connected.

3. The homotopy that is stationary on $S^1 \times \{0\}$ is for example $F\colon S^1 \times I \times I \to S^1 \times I$, $F(\vartheta,s,t) = (\vartheta + 2\pi s(1-t), s)$ (where $\vartheta \equiv e^{2\pi i\vartheta}$).
Suppose that there is a homotopy between $f(\vartheta,s)$ and the identity map that is stationary on $S^1 \times \{0\}$ and $S^1 \times \{1\}$. The projection of $f(\theta_0,s)$ onto S^1 (for a fixed θ_0)) as s varies represents a generator of $\pi_1(S^1) = \mathbb{Z}$, and the projection of the identity onto S^1 represents the zero element of $\pi_1(S^1)$. The homotopy would induce a homotopy between two different elements of π_1, which is a contradiction.