

SEMINAR IN TOPOLOGY (18.904)

HOMEWORK 3 – SOLUTIONS

1. **Note:** This is a trivial corollary of the (not so trivial) van Kampen's theorem.

We will prove the following: if U, V are open, simply connected with $x_0 \in U \cap V$ and $U \cap V$ path-connected, then $U \cup V$ is simply connected.

It is obvious that $U \cup V$ (the union of two path-connected sets with a non-empty intersection) is path-connected. Take a loop $f: I \rightarrow U \cup V$ with $f(0) = f(1) = x_0$. Then $f^{-1}(U) \cup f^{-1}(V)$ is an open covering of the compact metric space I , so there exists a Lebesgue number ε . If $0 = s_0 < s_1 < \dots < s_m = 1$ is a partition of I with $s_i - s_{i-1} < \varepsilon$, then each interval $[s_{i-1}, s_i]$ is mapped by f to either U or V (or both) – this implies in particular that $f(s_i) \in U \cap V$ for all i . Let f_i be the path f restricted to $[s_{i-1}, s_i]$. Since $U \cap V$ is path connected, we may choose a path g_i in $U \cap V$ from x_0 to $f(s_i)$. But then the loop

$$(f_1 \cdot \bar{g}_1) \cdot (g_1 \cdot f_2 \cdot \bar{g}_2) \cdots (g_{m-1} \cdot f_m)$$

is homotopic to f and is a composition of loops each lying in either U or V . Since U and V are simply connected, all these loops are null-homotopic, and hence their composition is null-homotopic. This implies that f is homotopic to the constant map c_{x_0} , and $U \cup V$ is simply connected.

If $\{U_1, \dots, U_n\}$ is a finite family of open simply connected sets with $U_i \cap U_j$ path-connected and $x_0 \in U_1 \cap \dots \cap U_n$, then we prove by induction that $U_1 \cup \dots \cup U_n$ is simply connected. We have proved the case for $n = 2$; and $V = U_{n-1} \cup U_n$ is simply connected, open, contains x_0 , and $V \cap U_i = (U_{n-1} \cap U_i) \cup (U_n \cap U_i)$ is path-connected as the union of path-connected sets with a non-empty intersection. By induction, $U_1 \cup \dots \cup U_{n-2} \cup V = U_1 \cup \dots \cup U_n$ is simply connected.

Now take X with the properties (a), (b) and (c). Since X is the union of path-connected sets with non-empty intersection, X is path-connected. Any loop $f: I \rightarrow X$ is contained in a finite subcover, e.g. $f(I) \subseteq U_1 \cup U_2 \cup \dots \cup U_n$, which is simply connected, and therefore f is null-homotopic.

2. Let N be the north pole of S^n , and S the south pole. Take $U = S^n \setminus \{N\}$, $V = S^n \setminus \{S\}$; then $U \approx V \approx \mathbb{R}^n$ are contractible (hence simply connected), $U \cap V \approx S^{n-1} \times (0, 1)$ is non-empty and path-connected (since $n \geq 2$), and therefore $\{U, V\}$ is the required open covering of S^n .