SEMINAR IN TOPOLOGY (18.904)

HOMEWORK 4 – SOLUTIONS

1. (a) Since $\mu i = \mu j = \mathrm{id}$, we have $\mu_*(i_*(\alpha)j_*(\beta)) = \mu_*(i_*(\alpha))\mu_*(j_*(\beta)) = \alpha\beta$. (b) Since $i_*\colon \pi_1(G,e) \to \pi_1(G \times G,(e,e)) = \pi_1(G,e) \times \pi_1(G,e)$ maps into $\pi_1(G,e) \times \{0\}$ and j_* maps into $\{0\} \times \pi_1(G,e)$, we have $i_*(\alpha)j_*(\beta) = j_*(\beta)i_*(\alpha)$ and by (a) $\alpha\beta = \mu_*(i_*(\alpha)j_*(\beta)) = \mu_*(j_*(\beta)i_*(\alpha)) = \beta\alpha$.

2. Let a, b be paths in G. Then
$$H\colon I \times I \longrightarrow G,$$
$$H(s,t) = \begin{cases} \mu(a(s(2-t)), b(st)) & : \quad 0 \le s \le 1/2 \\ \mu(a((1-t)+st), b((2s-1)(1-t)+st)) & : \quad 1/2 \le s \le 1 \end{cases}$$
defines a continuous function (because both definitions give $\mu(a(1-t/2), b(t/2))$ for $s = 1/2$), clearly we have $H(0,t) = H(1,t) = e$. In addition,
$$H(s,0) = \begin{cases} a(2s) & : \quad 0 \le s \le 1/2 \\ b(2s-1) & : \quad 1/2 \le s \le 1 \end{cases},$$
in other words, $H(s,0) = (a \cdot b)(s)$ and $H(s,1) = \mu(a(s), b(s))$.
We have proved that $[a \cdot b] = [\mu(a,b)]$. Then $[f(\gamma(s)) \cdot \gamma(s)] = [\mu(f(\gamma(s)), \gamma(s))] = [\gamma(s) \cdot f(\gamma(s))] = [\mu(\gamma(s), f(\gamma(s)))][e]$ and so $f_*([\gamma]) = [\gamma]^{-1}$.

3. By Proposition 1.17, if A is a retract of X, then the inclusion map $i : A \to X$ induces an injective homomorphism $\pi_1(A) \to \pi_1(X)$. In (a), the existence of a retract would imply the existence of an injection $\mathbb{Z} \to 0$ (which is obviously impossible), and in (b) the existence of an injection $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ (which is also impossible – if $\varphi(1,0) = a$ and $\varphi(0,1) = b$ then $\varphi(-b,a) = 0$).