SEMINAR IN TOPOLOGY (18.904)

HOMEWORK 5 – SOLUTIONS

1. Let $Y_k = \cup_{i=1}^k X_i$. Obviously Y_1 is simply connected, we will prove by induction that so is Y_k.
   The intersection $X_{k+1} \cap Y_k = \cup_{i=1}^k (X_{k+1} \cap X_i)$ is the union of nonempty convex (hence path-connected) sets with non-empty intersections $((X_{k+1} \cap X_i) \cap (X_{k+1} \cap X_j) = X_{k+1} \cap X_i \cap X_j)$, so it is path-connected. Since both X_{k+1} and Y_k are simply connected, $X_{k+1} \cup Y_k$ is simply-connected by van Kampen's theorem.

2. We can deformation retract the space X to S^{2n} with $2n$ points removed, which is homeomorphic to $\mathbb{R}^3$ with $2n-1$ points removed. Obviously, that means that
$$\pi_1(X) = \underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{2n-1}.$$

3. We use van Kampen's theorem: let A (respectively, B) be (the open neighborhood of) the first (respectively, the second) torus. Then $\pi_1(A) = \langle a, b | aba^{-1}b^{-1} \rangle$, $\pi_1(B) = \langle c, d | cdc^{-1}d^{-1} \rangle$, so
$$\pi_1(X) = \pi_1(A) * \pi_1(B) / \langle bd^{-1} \rangle = \langle a, b, c, d | aba^{-1}b^{-1}, cdc^{-1}d^{-1}, bd^{-1} \rangle =$$
$$= \langle a, b, c | aba^{-1}b^{-1}, bcb^{-1}c^{-1} \rangle.$$

4. Choose one of the vertices, and deformation retract the three adjoining edges to a point. We get $S^1 \vee S^1 \vee S^1$, with fundamental group $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.

5. Let $G_1 = \langle S_1 | R_1 \rangle$ and $G_2 = \langle S_2 | R_2 \rangle$. Then
$$G_1 \times G_2 = \langle S_1 \cup S_2 | R_1 \cup R_2 \cup \{ s_1 s_2 s_1^{-1} s_2^{-1} \colon s_1 \in S_1, s_2 \in S_2 \} \rangle,$$
$$G_1 * G_2 = \langle S_1 \cup S_2 | R_1 \cup R_2 \rangle$$
and
$$G_1 / [G_1, G_1] = \langle S_1 | R_1 \cup \{ ss's^{-1}s'^{-1} \colon s, s' \in S_1 \} \rangle.$$