18.904 HOMEWORK FOUR
DUE 3-8-2005

Complete the following problems. You are encouraged to work with other students in the class on the problems. Please write up your own solutions. Each problem is worth 10 points.

(1) Assume that G is a topological space, $\mu : G \times G \to G$ is a continuous map, and $e \in G$ is such that the following conditions hold: For any $x \in G$, $\mu(x, e) = \mu(e, x) = x$. Let $i : G \to G \times G$ and $j : G \to G \times G$ be defined by $i(x) = (x, e)$ and $j(x) = (e, x)$ for any $x \in G$.

(a) Prove that for any $\alpha, \gamma \in \pi_1(G, e)$, $\mu_*(i_*(\alpha)j_*(\beta)) = \alpha\beta$.

(b) Deduce that $\pi_1(G, e)$ is an abelian group.

An important example of this is when G is a *topological group*: G is a group that is also a topological space such that the group operation $G \times G \to G$ and inverse operation $G \to G$ are continuous maps. Some examples of topological groups are:

- $\mathbb{R}^m$ with addition as its group operation
- the general linear group, $GL(n, \mathbb{R})$ which is defined as the set of all *invertible* $n \times n$ matrices with real entries and matrix multiplication as its group operation. Note that this group is viewed as a topological space by viewing $GL(n, \mathbb{R})$ as a subset of the Euclidian space $\mathbb{R}^{n \times n}$.
- $O(n), U(n) \subset GL(n, \mathbb{R})$, the orthogonal and unitary groups.
  (Note that all of these examples are actually examples of *Lie Groups*, i.e. a group G for which G is a "smooth" manifold and the group and inverse operations are "smooth" maps.)

  Hence we have proved that all of the above spaces have abelian fundamental groups!

(2) Let G, e and μ be as in (1). Assume in addition that there exists a continuous map $f : G \to G$ such that $\mu(f(x), x) = \mu(x, f(x)) = e$ for any $x \in G$. Prove that for any $\gamma \in \pi_1(G, e)$, $f_*(\gamma) = \gamma^{-1}$.

Note that if G is a topological group then the inverse operation gives such a map f. Therefore for any of the examples above we know that $f_*(\gamma) = \gamma^{-1}$ where f is the given inverse operation.

(3) Exercise 16(a,b), p.39