18.904 HOMEWORK ONE
DUE 2-15-2005

Complete the following problems. You are encouraged to work with other students in the class on the problems. Please write up your own solutions. Each problem is worth 10 points.

(1) We recall a few definitions (these are all taken from Munkres book "Topology", pp. 135–143) about the quotient topology. Recall that a surjective map $p : X \to Y$ of topological spaces is a *quotient map* provided a subset U is open in Y if and only if $p^{-1}(U)$ is open in X. Now if $p : X \to S$ is a surjective map from a topological space X to a SET S then there is a unique topology on S which will make p a quotient map; $A \subset S$ is open if and only if $p^{-1}(A) \subset X$ is open.

Now let X be a topological space and X^* be a partition of X into disjoint subsets whose union is X; $X^* = \{S_i | i \in I\}$ such that $S_i \cap S_j = \emptyset$ if $i \neq j$ and $\cup_{i \in I} S_i = X$. There is a surjective map $p : X \to X^*$ defined by $p(x) = S_i$ if $x \in S_i$. Let X^* have the quotient topology induced by p then X^* is called a *quotient space* of X or is sometimes is called an *identification space* (since all the point in a set S_i are identified to a single point). Moreover, if $\sim$ is an equivalence relation on X then we can partition X into subsets by saying $x \in X$ and $y \in X$ are in the same subset if $x \sim y$. In this case, we may write $X/\sim$ instead of X^* for the quotient topology induced by $\sim$.

a. Define the equivalence relation $\sim$ on $\mathbb{R}^2$ by $(x, y) \sim (x', y')$ if $x' - x \in \mathbb{Z}$ and $y' - y \in \mathbb{Z}$. Prove that $\mathbb{R}^2$ is homeomorphic to $S^1 \times S^1$ (with the usual topology) (you may wish to use Theorem 11.1 or Theorem 11.2 in Munkres).

b. Define the equivalence relation $\sim$ on $\mathbb{R}^2$ by $(x, y) \sim (x', y')$ if $x^2 + y^2 = (x')^2 + (y')^2$. What familiar space is $X/\sim$ homeomorphic to? Please justify your response.

(2) Exercise 1, p. 18 (unless otherwise noted, these exercises will be from Hatcher)

(3) Exercise 9, p. 19