

18.905 EXAM - FALL 2004

NAME: Solutions

Instructions: Don't forget to write your name above!!! Complete each of the following four problems. You may not use any notes or books. Please show all of your work. If you use a theorem, state the theorem you are using.

1. For each of the following six statements, either give specific examples of (connected) topological spaces X and Y which satisfy the statement or explain (in one or two sentences) why no such examples exist. In this problem, by homology groups or cohomology groups, I mean homology groups with \mathbb{Z} coefficients and cohomology groups with \mathbb{Z} coefficients.

- (a) X is homotopy equivalent to Y , but X is not homeomorphic to Y .

One example: X is the letter X and Y is the letter Y .

Another example: $X = D^2$, $Y = \text{point}$.

- (b) $\pi_1(X) \cong \pi_1(Y)$ but $H_1(X) \not\cong H_1(Y)$.

No such example exists if X and Y are path-connected since $H_1(X)$ is the abelianization of $\pi_1(X)$ (similarly for Y).

- (c) $H_p(X) \cong H_p(Y)$ for all p but $\pi_1(X) \not\cong \pi_1(Y)$.

One such example: $X = S^1 \vee S^1 \vee S^2$ and $Y = S^1 \times S^1$.

Another such example: X and Y are knot complements. For example, let $X = S^3 - K$ where K is the trefoil knot and $Y = S^3 - T$ where T is the trivial knot.

- (d) X and Y have isomorphic homology groups (in all dimensions) but non-isomorphic cohomology groups (in some dimension).

No such examples exist since by the Universal Coefficient Theorem, $H^n(X; \mathbb{Z})$ is completely determined by $H_n(X; \mathbb{Z})$ and $H_{n-1}(X; \mathbb{Z})$ (similarly for Y).

- (e) X and Y have isomorphic homology and cohomology groups (in all dimensions), and $\pi_1(X) \cong \pi_1(Y)$, but X is not homotopy equivalent to Y .

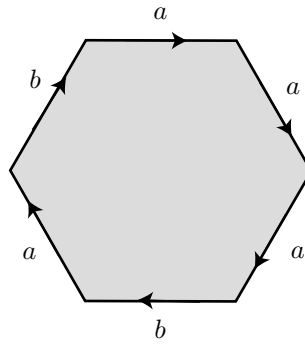
$X = \mathbb{C}P^2$ and $Y = S^2 \vee S^4$. These spaces have the same homology and cohomology groups and both have $\pi_1 = \{1\}$ but they have non-isomorphic cohomology rings so they are not homotopy equivalent.

- (f) There is a continuous map $f : X \rightarrow Y$ inducing isomorphisms on all of the cohomology groups (i.e. $f^* : H^p(Y) \xrightarrow{\cong} H^p(X)$ for all p) but X and Y do not have isomorphic cohomology rings (with \mathbb{Z} coefficients).

No such examples exist since f^* is a ring homomorphism (it preserves the cup product)!

Note: The spaces $X = \mathbb{C}P^2$ and $Y = S^2 \vee S^4$ (from above) have (abstractly) isomorphic homology and cohomology groups. However, since their cohomology rings are distinct, there is no map $f : \mathbb{C}P^2 \rightarrow S^2 \vee S^4$ (or $f : S^2 \vee S^4 \rightarrow \mathbb{C}P^2$) which induces these isomorphisms.

2. Let X be the following identification space.



- (a) Describe a CW structure for X . Using this structure, give the cellular chain complex for X .

X has the structure of a CW complex with 1 0-cell v , 2 1-cells a and b and 1 2-cell D . The boundary of D attaches to the 1-skeleton of X by the word a^3bab .

The cellular chain complex for this structure is

$$0 \rightarrow \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^2 \xrightarrow{\partial_1} \mathbb{Z} \rightarrow 0$$

where $\partial_2(1) = (2, 4)$ and $\partial_1 = 0$.

- (b) Using part (a), compute $H_i(X; \mathbb{Q})$ and $H_i(X; \mathbb{Z}_2)$ for all i .

Since X is connected and has no cells n -cells for $n \geq 3$, $H_0(X; G) \cong G$ and $H_i(X; G) = 0$ for $i \neq 0, 1, 2$.

To compute $H_i(X; G)$ we just need to tensor the above chain complex with G and take the homology. We identify $\mathbb{Z}^m \otimes G$ with G .

Let $G = \mathbb{Q}$ then ∂_2 is injective and hence the image of ∂_2 spans a 1-dimensional subspace of $C_1(X; G) = G \oplus G$. Thus, $H_1(X; \mathbb{Q}) \cong \mathbb{Q}$ and $H_2(X; \mathbb{Q}) = 0$.

Let $G = \mathbb{Z}_2$ then $\partial_2 = 0$. Hence, $H_1(X; \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $H_2(X; \mathbb{Z}_2) = \mathbb{Z}_2$.

3. Let M be a closed, connected, orientable 4-dimensional manifold with $\pi_1(M) \cong \mathbb{Z}_3 * \mathbb{Z}_3$ and $\chi(M) = 5$.

(a) Compute $H_p(M; \mathbb{Z})$ for all p .

We are always using \mathbb{Z} coefficient group in this problem.

Since M is a closed, orientable, connected 4-manifold, $H_0(M) \cong H_4(M) \cong \mathbb{Z}$.

$H_1(M)$ is the abelianization of $\pi_1(M)$ hence $H_1(M) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$.

By Poincaré Duality and the Universal Coefficient Theorem for Cohomology, $H_2(M) \cong H^2(M) \cong \text{Hom}(H_2(M), \mathbb{Z}) \oplus \text{Ext}(H_1(M), \mathbb{Z}) \cong \mathbb{Z}^{\beta_2(M)} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$.

By Poincaré Duality and the Universal Coefficient Theorem for Cohomology, $H_3(M) \cong H^1(M) \cong \text{Hom}(H_1(M), \mathbb{Z}) \oplus \text{Ext}(H_0(M), \mathbb{Z}) \cong 0$.

We can compute $\beta_2(M)$ since we know the Euler characteristic of M :

$5 = \chi(M) = \sum_{i=0}^4 (-1)^i \beta_i(M) = 2 + \beta_2(M)$ hence $\beta_2(M) = 3$.

Therefore $H_2(M) \cong \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$.

(b) Prove that M is not homotopy equivalent to any CW complex with no 3-cells.

Suppose that X is a CW complex with no 3-cells. The cellular chain complex of X has no 3-cells hence $\text{im}(\partial_3) = 0$. Therefore $H_2(X) = \ker(\partial_2) \subset C_2(X)$ is a subgroup of a free abelian group hence must be free abelian. Now, if $f : M \rightarrow X$ is a homotopy equivalence then f_* must be an isomorphism on all of homology (in particular, $H_2(M) \cong H_2(X)$). Since $H_2(X)$ is free abelian and $H_2(M)$ is not free, they cannot be isomorphic. Thus, M cannot be homotopy equivalent to any CW complex with no 3-cells.

4.

- (a) State the cohomology ring of
- $\mathbb{R}P^n$
- with
- \mathbb{Z}_2
- coefficients.

$$H^*(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha] / \langle \alpha^{n+1} \rangle \text{ and } |\alpha| = 1.$$

- (b) Recall that
- $\mathbb{R}P^{n-1}$
- can be considered as a subspace of
- $\mathbb{R}P^n$
- by the standard inclusion map
- $i : \mathbb{R}P^{n-1} \rightarrow \mathbb{R}P^n$
- . Viewing
- $\mathbb{R}P^{n-1} \subset \mathbb{R}P^n$
- in this way, show that, if
- $n \geq 2$
- ,
- $\mathbb{R}P^{n-1}$
- is not a retract of
- $\mathbb{R}P^n$
- .

Let α be the generator of $H^1(\mathbb{R}P^n; \mathbb{Z}_2)$, then $\alpha^n \neq 0$. Recall that the inclusion map $i : \mathbb{R}P^{n-1} \rightarrow \mathbb{R}P^n$ induces an isomorphism $i^* : H^1(\mathbb{R}P^n; \mathbb{Z}_2) \rightarrow H^1(\mathbb{R}P^{n-1}; \mathbb{Z}_2)$ for $n \geq 2$. Thus $\beta = i^*(\alpha)$ generates $H^1(\mathbb{R}P^{n-1}; \mathbb{Z}_2)$. In particular, $\beta^n = 0$.

Suppose that $r : \mathbb{R}P^n \rightarrow \mathbb{R}P^{n-1}$ is a retract. By definition, $r \circ i = \text{id}_{\mathbb{R}P^{n-1}}$. Hence, we have $i^* \circ r^* = \text{id}_{H^1(\mathbb{R}P^{n-1}; \mathbb{Z}_2)}$ for all i . In particular, we see that $i^*(r^*(\beta)) = \beta$. Since i^* is injective and $i^*(\alpha) = \beta$, we conclude that $r^*(\beta) = \alpha$. Therefore

$$\alpha^n = r^*(\beta)^n = r^*(\beta^n) = r^*(0) = 0$$

which is a contradiction. Thus, $\mathbb{R}P^{n-1}$ is not a retract of $\mathbb{R}P^n$!