18.905 HOMEWORK FIVE
DUE 10-13-2004
IN CLASS OR TURNED IN TO THE
UNDERGRADUATE MATH OFFICE 2-108 BY 3PM*

Reading Assignment for the week:

Mon 10-11: holiday, no classes!
Wed 10-13: pp. 149-151 (Mayer-Vietoris and examples)

Complete the following problems. You are encouraged to work with other students in the class on the problems. You may also consult Hatcher's book on Algebraic Topology. However, you must write up your own solutions. Please use complete sentences when writing up your solutions.

1. Without looking at the proof of Theorem 2.16 (a.k.a. the "Zig-Zag Lemma") in the book, prove the following 4 statements where i_*, j_* and ∂ are as defined on p. 116 (or as defined in class). These statements were not proved in class and are the remaining verifications we need in order to complete the proof of the "Zig-Zag Lemma."
   (i) $\operatorname{im} i_* \subset \ker j_*$
   (ii) $\operatorname{im} j_* \subset \ker \partial$
   (iii) $\operatorname{im} \partial \subset \ker i_*$
   (iv) $\ker \partial \subset \operatorname{im} j_*$

2. Exercise 15, p. 132

3. (i) Exercise 17(b), p.132. However, only compute $H_0(X, A)$, $H_1(X, A)$, $H_0(X, B)$ and $H_1(X, B)$. You will need to use that H_1 is the abelianization of π_1 to compute $H_1(X)$. You should note that although, $H_1(A) = H_1(B)$, the relative homology groups are not the same.
   (ii) Assuming that $H_n(S^1) = 0$ for $n \geq 2$, compute the relative homology groups $H_n(X, \partial X)$ for all $n \geq 0$ when X is an annulus ($X = I \times S^1$ and $\partial X = \{0\} \times S^1 \sqcup \{1\} \times S^1 \subset X$).

4. The **Five-Lemma** states: Suppose the following diagram of abelian groups commutes.

$$\begin{array}{ccccccccc}
A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & D & \xrightarrow{\delta} & E \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\
A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \xrightarrow{\gamma'} & D' & \xrightarrow{\delta'} & E'
\end{array}$$

   If the two rows are exact and $f, g, i,$ and j are isomorphisms then h is an isomorphism.
   (i) Prove the Five-Lemma without looking at the proof in the book.
   (ii) What are the minimal conditions one must put on f, g, i and j to ensure that h is surjective?
   (iii) What are the minimal conditions one must put on f, g, i and j to ensure that h is injective?

5. Let $f : S^1 \times D^2 \to \mathbb{R}^3$ be a topological embedding. The image $K = f(S^1 \times \{0\})$ is called a *(tame) knot* in $\mathbb{R}^3$. The complement $\mathbb{R}^3 - K$ is called the *knot complement.* Assume for this problem that $H_n(S^1) = 0$ for $n \geq 2$, $H_2(S^1 \times S^1) = \mathbb{Z}$, and $H_m(S^1 \times S^1) = 0$ for $m \geq 3$ (which you should know to be true in simplicial homology). Prove that $H_1(\mathbb{R}^3 - K) \cong Z$, $H_2(\mathbb{R}^3 - K) \cong \mathbb{Z}$ and

† Late homeworks will not be graded and will receive at most 50% of the total grade.
* If you do not turn in your homework in class, make sure that your homework is turned into the Undergraduate Math Office 2-108 before 3pm on Wed 10-13-2004.

$H_n(\mathbb{R}^3 - K) = 0$ for $n \geq 3$. Hence, the homology of a knot complement independent of the embedding of the knot. Thus, the situation in homology is in stark contrast to the situation for π_1 since $\pi_1(\mathbb{R}^3 - K)$ is highly non-trivial and can in theory distinguish most knots! To do this problem, you will need to use the Exactness Axiom (the existence of a long exact sequence of a pair), the Excision Axiom (Theorem 2.20), and the Homotopy Axiom for a map of pairs (Proposition 2.19).

Have a nice holiday!!!