18.905 HOMEWORK FOUR
DUE 10-06-2004 BEFORE 3PM†

Reading Assignment for the week (some of the reading was listed on last week's reading assignment):

Wed 9-29:
- pp. 166-168 (Homology and the Fundamental Group)
- The section of p. 110 on Reduced Homology Groups
- pp. 111–113 (Homotopy invariance)
- pp. 113–114 stop before Theorem 2.13 (Exact Sequences)

Mon 10-4:
- pp. 114–118 (Relative Homology Groups)

Wed 10-6:
- pp. 119–120 (The Statement of the Excision Theorem and Barycentric Subdivision of Simplices)

Complete the following problems. You are encouraged to work with other students in the class on the problems. You may also consult Hatcher's book on Algebraic Topology. However, you must write up your own solutions. Please use complete sentences when writing up your solutions.

1. Let $f : X \to Y$ and $g : Y \to Z$. Prove directly from the definitions that $f_* \circ g_* = (f \circ g)_*$ where $f_* : H_n(X) \to H_n(Y)$, $g_* : H_n(Y) \to H_n(Z)$, and $(f \circ g)_* : H_n(X) \to H_n(Z)$. Prove directly from the definitions that $(id_X)_* = id_{H_n(X)}$ where $id_X : X \to X$ is the identity map on X and $id_{H_n(X)} : H_n(X) \to H_n(X)$ is the identity map on $H_n(X)$.

2. Let X be a non-empty topological space with $n < \infty$ path-connected components. Prove that $\widetilde{H}_0(X)$ is isomorphic to $\mathbb{Z}^{n-1}$. Find a basis for $\widetilde{H}_0(X)$ and prove that it is a basis.

3. Let X and Y be topological spaces and $f : X \to Y$ be a constant map. Prove that $f_* = 0$ on $H_n(X)$ with $n \geq 1$ (i.e. $f_*(\alpha) = 0$ for all $\alpha \in H_n(X)$).

4. Suppose
$$\cdots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \to \dots$$
is a long exact sequence of abelian groups. Show that there is a short exact sequence of the form
$$0 \to \mathrm{cok}(\alpha_{n+2}) \xrightarrow{\overline{\alpha}_{n+1}} A_n \xrightarrow{\alpha'_n} \ker(\alpha_{n-1}) \to 0.$$
Note that you first need to define the maps $\overline{\alpha}_{n+1}$ and α'_n and show that they are well-defined.

5. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a short exact sequence of abelian groups.
   (i) Suppose there exists a homomorphism $s : C \to B$ such that $\beta \circ s = \mathrm{id}_C : C \to C$. This type of short exact sequence is called **split**. The map s is called a **splitting**. Show that $B \cong A \oplus C$.
   (ii) Suppose that $C = \mathbb{Z}^m$. Show that there is a splitting $s : \mathbb{Z}^m \to B$.
   (iii) Show that if X is a non-empty topological space then $H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$.

† Late homeworks will not be graded and will receive at most 50% of the total grade.