18.905 HOMEWORK ONE
DUE 9-15-2004

Reading Assignment for the week of 9-8 through 9-15:

Chapter 0, pp. 1–13
Chapter 1, pp. 21–28
†Chapter 1, statement and proof of Theorems 1.8-1.10 on pp. 31-33
Chapter 1, pp. 34-37

Complete the following problems. You are encouraged to work with other students in the class on the problems. You may also consult Hatcher's book on Algebraic Topology. However, you must write up your own solutions. Please use complete sentences when writing up your solutions.

1. Let $f : I \to I$ be any map such that $f(0) = 0$ and $f(1) = 1$ where $I = [0, 1]$.
   i) Show that f is homotopic (rel $\{0, 1\}$) to the identity map on I, $\mathrm{id}_I : I \to I$.
   ii) Let $g : I \to X$ be any map and f be as above. Recall that $g \circ f$ is called a reparametrization of g. Using part i, show that $g \circ f$ is homotopic (rel $\{0, 1\}$) to g.

2. We recall the definition of the "connected sum" of surfaces. Let S_1 and S_1 be disjoint connected surfaces. Their *connected sum*, denoted $S_1 \# S_2$, is formed by cutting a circular hole in each surface, and then gluing the two surfaces together along the boundaries of the holes. More precisely, let $D_i \subset S_i$ be closed disks in S_i. Choose a homeomorphism $h : \partial D_1 \to \partial D_2$. Then $S_1 \# S_2$ is the quotient space of $(S_1 - \mathrm{int}(D_1)) \sqcup (S_2 - \mathrm{int}(D_2))$ obtained by identifying the points x and $h(x)$ for all points $x \in \partial D_1$. One can show that the topological type of $S_1 \# S_2$ does not depend on the choice of the disk D_i or homeomorphism h.

   Let Σ_g be the quotient space of the $4g$-gon obtained by identifying (directed) edges in pairs as follows. We first start from any vertex of the $4g$-gon. We label the edges (moving clockwise) by the following word
$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$$
in the sense that we label the i^{th} edge with the i^{th} letter in the word. We label the i^{th} edge with an arrow pointing in the counterclockwise direction if the letter has a -1 exponent and an arrow pointing clockwise if the exponent is $+1$. For example, Σ_1 is the torus $S^1 \times S^1$ (see Figure 1). Also, see Hatcher's book on page 5 for an example when $g = 2$ and $g = 3$.

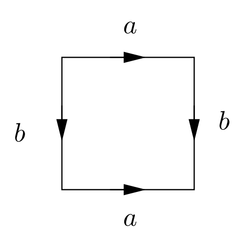

FIGURE 1. The torus

   Prove that Σ_g is homeomorphic to $\Sigma_1 \# \cdots \# \Sigma_1$ (g times). Your proof should primarily consist of cutting and pasting of planar diagrams along with the necessary explanations.
   Hint: Consider the picture of the 8-gon in Hatcher's book with labels a, b, c, and d. What spaces do you get if you cut the 8-gon in half, separating the a, b labels and the c, d labels?

†This part of the reading is not required but is recommended. It includes a few nice applications of π_1 to algebra and topology.

3. Recall that $\mathbb{C}\mathrm{P}(n)$ is the quotient space of $\mathbb{C}^{n+1} - \{(0, \ldots, 0)\}$ (with quotient map $q : \mathbb{C}^{n+1} - \{(0, \ldots, 0)\} \to \mathbb{C}\mathrm{P}(n)$) wherein a point $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} - \{(0, \ldots, 0)\}$ is identified with all points $(\lambda z_0, \ldots, \lambda z_n)$ for $\lambda \in \mathbb{C} - \{0\}$. We denote $q((\lambda z_0, \ldots, \lambda z_n))$ by $[z_0, \ldots, z_n]$. (Thus $[z_0, \ldots, z_n] = [\lambda z_0, \ldots, \lambda z_n]$ for $\lambda \neq 0$). Given $P = [z_0, \ldots, z_n]$, a choice of $(z_0, \ldots, z_n)$ representing P is called homogeneous coordinates for P.

i) Show that any point in $\mathbb{C}\mathrm{P}(n)$ has homogeneous coordinates such that $\sum_{i=0}^{n} ||z_i||^2 = 1$, i.e. $(z_0, \ldots, z_n) \in S^{2n+1} \subset \mathbb{R}^{2(n+1)} = \mathbb{C}^{n+1}$. Thus the map q factors as

$$\mathbb{C}^{n+1} - \{(0, \ldots, 0)\} \xrightarrow{p} S^{2n+1} \xrightarrow{\pi} \mathbb{C}\mathrm{P}(n).$$

Conclude that $\mathbb{C}\mathrm{P}(n)$ is compact.

ii) For each $i \in \{0, \ldots, n\}$, let $U_i = \{[z_0, \ldots, z_n] \in \mathbb{C}\mathrm{P}(n) | z_i \neq 0\}$. Show that the condition $z_i \neq 0$ is independent of the choice homogeneous coordinates. Show that U_i is an open set homeomorphic to $\mathbb{C}^n$. Show that $\mathbb{C}\mathrm{P}(n)$ is a $2n$-dimensional manifold (see p. 231 in Hatcher for the definition of manifold). Show that $\mathbb{C}\mathrm{P}(n) - U_i$ is homeomorphic to $\mathbb{C}\mathrm{P}(n-1)$.