18.905 HOMEWORK TEN
DUE 11-17-2004 BY 3PM

Reading Assignment for the week:

Mon 11-15: pp. 212–217, Cohomology Ring
Wed 11-17: (more cohomology ring) and pp. 230–233, Poincare Duality

Complete the following problems. You are encouraged to work with other students in the class on the problems. You may also consult Hatcher's book on Algebraic Topology. However, you must write up your own solutions. Please use complete sentences when writing up your solutions.

1. Exercise 1, p. 267

2. A topological space X is said to be a *rational homology ball* if $H_i(X;\mathbb{Q}) \cong H_i(B^k;\mathbb{Q})$ for all i where B^k is a k-dimensional ball. Note: The homology of B^k is independent of k so this definition makes sense.
   (a) For which $1 \leq m \leq \infty$ is $\mathbb{R}\mathrm{P}^m$ a rational homology ball?
   A topological space X is said to be a *rational homology n-sphere* if $H_i(X;\mathbb{Q}) \cong H_i(S^n;\mathbb{Q})$ for all i where S^n is the n-dimensional sphere.
   (b) For which $1 \leq m \leq \infty$ is $\mathbb{R}\mathrm{P}^m$ a rational homology n-sphere?
   (c) For $1 \leq m \leq \infty$, use the Universal Coefficient Theorem for Homology to compute $H_i(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2)$ (do not use the use result from problem 3). Are these the same as the cohomology groups of $\mathbb{R}\mathrm{P}^m$ with coefficient group $\mathbb{Z}_2$ that you got in the last HW assignment?

3. Let C be a free chain complex. Show that if $H_i(C)$ is finitely generated for all i, then
$$H_p(C;\mathbb{Z}_n) \cong H^p(C;\mathbb{Z}_n).$$
   Note: This isomorphism is not natural.

4. Let $f : S^2 \to T$ be continuous where T is the torus. Show that $f^* : H^2(T;\mathbb{Z}) \to H^2(S^2;\mathbb{Z})$ is trivial, and conclude that $f_* : H_2(S^2) \to H_2(T)$ is trivial. What can you say about a continuous map $g : T \to S^2$.

5. Compute the cohomology ring of the Klein bottle, K, with $\mathbb{Z}_2$ coefficients. Use this to show that if $f : K \to T$ then $f^* : H^2(T;\mathbb{Z}_2) \to H^2(K;\mathbb{Z}_2)$ is trivial.
   Note: You can also use the cohomology rings to show that a map $f : S^2 \to K$ (respectively $f : T \to K$) always induced a trivial map $f^* : H^2(K;\mathbb{Z}_2) \to H^2(S^2;\mathbb{Z}_2)$ (respectively $f^* : H^2(K;\mathbb{Z}_2) \to H^2(T;\mathbb{Z}_2)$)!

† Late homeworks will not be graded and will receive at most 50% of the total grade.