18.905 HOMEWORK TWELVE
OPTIONAL

NOTE: There is an in-class exam on the last day of class (Wednesday 12-08)!

Reading Assignment:

pp. 239–242, The Duality Theorem (up to Cohomology with Compact Support)
pp. 254, The statement of Theorem 3.43 (you will need to use this in Exercise 1)

Optional Reading (I would recommend that graduate students and future graduate students learn this material ... but not necessarily before the exam!!!):

pp. 242–245, Cohomology with Compact Support and Duality for Noncompact Manifolds
pp. 253–257, Other Forms of Duality
pp. 275, (Theorem 3B.6), The Topological Künneth Formula

Complete the following problems. You are encouraged to work with other students in the class on the problems. You may also consult Hatcher's book on Algebraic Topology. However, you must write up your own solutions. Please use complete sentences when writing up your solutions.

1. Suppose M is a compact, connected n-dimensional manifold (possibly non-orientable) with non-empty connected boundary ∂M. Show that the Euler characteristic of ∂M is even. Conclude that the non-orientable surface of genus g with g odd (the connected sum of g copies of $\mathbb{R}P^2$) is not the boundary of any compact 3-dimensional manifold (orientable or non-orientable). Note that the Klein bottle (non-orientable surface of genus 2) is the boundary of a non-orientable 3-dimensional manifold, the "solid Klein bottle."

   Hint: You will need to use "Lefschetz Duality" (Theorem 3.43) and the long exact sequence of a pair.

2. Let M and N be connected, closed (compact with no boundary), orientable n-dimensional manifolds and $f : M \to N$ be a continuous map. Suppose the degree of f is non-zero. Show that $\beta_i(M) \geq \beta_i(N)$ for all i.

   See problem 7 on p. 258 in Hatcher for the definition of the degree of f. Recall that $\beta_i(M)$ is the i^{th} Betti number of M which is finite since the homology groups of a compact n-dimensional manifold are finitely generated.