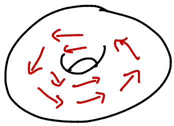
Day 1 - Introduction

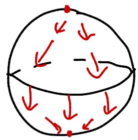
Let M be a smooth closed manifold.

Q. Does M admit a nowhere zero vector field?

$S^1 \times S^1$



Yes



S^2

No

Recall that a surface S has a nowhere zero vector field $\Leftrightarrow \chi(S) = 0$.

More generally, if M^n is a finite CW-complex then we can define

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \, (\# \text{ of cells of dim } i)$$

<u>Thm</u>: $\chi(M) = \#$ of zeros of vector field (counted w/ sign)

This related to homology/cohomology:

<u>Thm</u>: $\chi(M) = \sum_{i=0}^{n} (-1)^i \text{ rank of } H_i(M)$

Recall, the tangent space of M is

$$TM = \bigcup_{p \in M} T_pM$$

$$= \bigcup_{p \in M} \{(p,v) \mid p \in M,\ v \text{ is a vector tangent to } M \text{ at } p\}$$

Let $Z(M) = \{(p,\vec{0}) \mid p \in M\} \subset TM$ be the zero set and $Z(M)^+$ be a pushoff of $Z(M)$ into TM.

<u>Thm</u>: $Z(M) \cap Z(M)^+ = \chi(M)$

Hence χ reflects an intersection #.

Let κ be the Gaussian curvature for any embedding of $M \subseteq \mathbb{R}^N$.

<u>Thm</u>: $\chi(M) = \frac{1}{\mathrm{vol}(S^n)} \int_M \kappa(x)\, d\mathrm{vol}_M$.

$\Rightarrow \chi$ is an integral of curvature.

From deRham cohomology, $\exists$ n-form Λ_M generating $H^n(M;\mathbb{R}) \cong \mathbb{R}$ s.t.

$$\chi(M) = \Lambda([M])$$

fundamental class in $H_n(M)$

hence we can view χ as a cohomology class Λ_M, called the Euler class.

More generally, using work of Stiefel, Whitney, Pontrjagin, and Chern we can general χ.

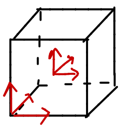
$$M \rightsquigarrow TM \text{, a rank } n \text{ vector bundle}$$

How much does this structure of TM tell us about M?

Q. Can we find k linearly independent (hence nowhere zero) vector fields on TM.

Thm (Stiefel): Every compact 3-mfld has a set of three nowhere zero vector fields that are everywhere linearly independent.

Ex: $T^3 = S^1 \times S^1 \times S^1$



Ex: S^3 = unit quaternions
(Lie group)

Can find a basis at identity.

Push around by left multiplication.

$$\Longrightarrow TM^3 \cong M^3 \times \mathbb{R}^3 \quad \text{(product bundle)}$$

To obstruct this, can use Stiefel-Whitney classes.

To any vector bundle of rank n, $\xi : E \xrightarrow{\pi} M$,
E locally looks like a product. For $p \in M$, $p \in U$ s.t. $\pi^{-1}(U) \cong U \times \mathbb{R}^n$. We get

$$w_i(\xi) \in H^i(M; \mathbb{Z}_2).$$

Thm: If $w_i(\xi) \neq 0$ then ξ cannot have $(n-i+1)$ everywhere linearly independent sections.

Pontrjagin/Thom Thm: A closed smooth manifold M^n is the boundary of a compact smooth manifold $\Leftrightarrow$ all the Stiefel-Whitney classes vanish.

Ex: $\mathbb{RP}^3 = \partial W^4$, what is W^4?

- w_1 obstructs M being orientable
- For a simply connected 4-mfld, $w_1 = w_2 = 0$ (called a spin 4-manifold) $\Leftrightarrow$ the intersection form is even.

Chern classes:

$$\text{complex v.b. } \xi \rightsquigarrow c_i(\xi) \in H^{2i}(M;\mathbb{Z})$$

Pontrjagin classes: can complexify a real vector bundle and consider its Chern classes.

$$\text{real v.b. } \xi \rightsquigarrow p_i(\xi) \in H^{4i}(M;\mathbb{Z}).$$

Can use the Pontrjagin classes to prove:

Thm (Milnor): $\exists$ triangulated 7-mfld which is homeomorphic (in fact PL-equiv) to S^7 but is not diffeomorphic to S^7.