Day 11

> Given G acting on F, and $P \xrightarrow{p} B$ a principal Gbundle $\rightsquigarrow$ fiber bundle with fiber F.

$$P \times_G F = (P \times F)/\sim \quad (x,f) \sim (xg, g^{-1}f) \ \forall g \in G$$

$[x,f]$ = equiv. class of (x,f)

$$\begin{array}{ccc} P \times_G F & & [x,f] \\ \downarrow q & & \downarrow \\ B & & p(x) \end{array}$$

well-defined since $p(xg) = p(x)$. $\quad (P/G = B)$

Fix $b \in B$, then $\exists$ charts

$$\varphi : U \times G \longrightarrow p^{-1}(U)$$

- $U \times F \xrightarrow{\alpha} (U \times G) \times_G F$ is a homeo with
  $(u,f) \longmapsto [(u,1), f]$

inverse: $[(u,g), f] \longmapsto (u, gf)$

well-defined?

$$[(u,g)h, h^{-1}f] = [(u,gh), h^{-1}f] \longrightarrow (u, gh \cdot h^{-1}f) = (u, gf)$$

Get charts for fiber bundle

$$\begin{tikzcd} U \times F \arrow[r, "\alpha"] \arrow[rr, bend right, "\psi"'] & (U \times G) \times_G F \arrow[r, "\varphi \times_G id"] & p^{-1}(U) \times_G F = q^{-1}(U) \end{tikzcd}$$

$$(u,f) \longmapsto [(u,1), f] \longmapsto [\varphi(u,1), f]$$

Note: $\varphi \times_G id$ is well-defined since

$$[(u,g), f] \longmapsto [\varphi(u,g), f]$$

$$[(u,g)h, h^{-1}f] \longmapsto [\varphi(u,gh), h^{-1}f]$$

since G act on right on $P \xrightarrow{p} B$

$$= [\varphi(u,g)h, h^{-1}f]$$

$$= [\varphi(u,g), f] \checkmark$$

<u>Transition functions</u> $\quad U = U_1 \cap U_2$

$$U \times F \xrightarrow{\psi_1} q^{-1}(U) \xrightarrow{\psi_2^{-1}} U \times F$$

$$(u,f) \longmapsto (\varphi_1(u,1), f) \longmapsto (u, \bar{\theta}_{12}(u) f)$$

Hence

$$\psi_2(u, \bar{\theta}_{12}(u) f) = (\varphi_2(u,1), \bar{\theta}_{12}(u) f) = (\varphi_2(u, \bar{\theta}_{12}(u)), f)$$

but also $\psi_2(u, \bar{\theta}_{12}(u,f)) = (\varphi_1(u,1), f)$

$$\Rightarrow \varphi_1(u,1) = \varphi_2(u, \bar{\theta}_{12}(u))$$

There are exactly the transition functions for $P \xrightarrow{p} B$!

$$\begin{array}{ccccc} U \times G & \xrightarrow{\varphi_1} & p^{-1}(U) & \xrightarrow{\varphi_2^{-1}} & U \times G \\ (u,1) & \longmapsto & & & (u, \theta_{12}(u)) \end{array}$$

so $\varphi_2(u, \theta_{12}(u)) = \varphi_1(u,1)$

Thus $\boxed{\bar{\theta}_{ij} = \theta_{ij}}$ $\forall\, i,j$.

The transition functions are the same!

The transition functions θ_{ij} satisfy the cocycle conditions so can build a fiber bundle using with fiber F with

$$\theta_{ij} : U_i \cap U_j \longrightarrow G \hookrightarrow \mathrm{Homeo}(F).$$
(or $\mathrm{Diffeo}(F)$)

We say that $q : P \times_G F \longrightarrow B$ is the fiber bundle associated to the principal bundle $p : P \to B$. via the action of G on F.

Suppose
$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow q \\ & & B \end{array}$$
is a fiber bundle with structure group G. Then can use transition functions $\theta_{ij} : U_i \cap U_j \longrightarrow G \hookrightarrow \mathrm{Homeo}(F)$ to build principal G-bundle. (Use $U \times G$ as building blocks instead of $U \times F$)

These are inverses up to <u>fiber bundle isomorphisms.</u>

<u>Def:</u> We say that two fiber bundles with structure group G and fiber F are isomorphic if $\exists$ morphism of fiber bundles that has an inverse.

i.e. have

$$\begin{tikzcd} E_1 \arrow[r, "\tilde{f}"] \arrow[d, "p_1"'] & E_2' \arrow[r, "\tilde{g}"] \arrow[d, "p_2"] & E_1 \arrow[d, "p_1"] \\ B_1 \arrow[r, "f"] & B_2 \arrow[r, "g"] & B_1 \end{tikzcd}$$

commute

- $\hat{f}, \tilde{g}$ are inverses (both directions)
  f, g " "

For every chart $\varphi_i : U_i \times F \longrightarrow p_i^{-1}(U)$ and $b \in B_1$,

$$\{b\} \times F \xrightarrow{\phi_1} p_1^{-1}(b) \xrightarrow{\tilde{f}} p_2^{-1}(f(b)) \xrightarrow{\phi_2^{-1}} \{f(b)\} \times F$$

is not just a homeo/diffeo but an element of G!

Sometimes may require base spaces to be same and $f = g = id_B$.