Day 12 - First Stiefel-Whitney class

Given v.b. $\xi : E \xrightarrow{\pi} B$, we associate

$$w_1(\xi) \in H^1(B;\mathbb{Z}_2)$$

with property that

$$w_1(\xi) = 0 \iff \xi \text{ is an orientable bundle.}$$

Define as follows:

$$H^1(B;\mathbb{Z}_2) \xrightarrow[\kappa]{\cong} \mathrm{Hom}_{\mathbb{Z}}(H_1(B),\mathbb{Z}_2) \xleftarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(\pi_1(B),\mathbb{Z}_2)$$

Hence an element $\alpha \in H^1(B;\mathbb{Z}_2)$ assigns $\mathbb{Z}_2$ to each loop in B.

<u>Lemma</u>(prove later): There are exactly 2 bundles of rank n over S^1.

- $S^1 \times \mathbb{R}^n$ (trivial bundle)
- Mobius bundle $\oplus\, \varepsilon^{n-1}$ (non-trivial)

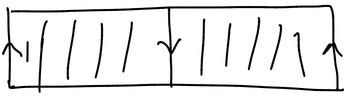
<u>Note</u>: MB not stably trivial

Define: Let $\gamma : S^1 \to B$ be a loop

$$\begin{array}{ccc} \gamma^*(\xi) & \longrightarrow & E \\ \downarrow & & \downarrow \pi \;= \xi \\ S^1 & \longrightarrow & B \end{array}$$

Define $\quad w_1(\xi)(\gamma) = \begin{cases} 0 & \text{if } \gamma^*(\xi) \text{ is trivial} \\ 1 & \text{if } \gamma^*(\xi) \text{ not trivial} \end{cases}$

Can show that this is additive since

 = trivial

If $\gamma \sim \delta$ then need to show pullback bundles are same. but then $\gamma\bar{\delta}$ is null-homotopic

and $\quad w_1(\xi)(\gamma\bar{\delta}) = w_1(\xi)(\gamma) + w_1(\xi)(\bar{\delta})$

so need to show for nullhomotopic $f : S^1 \to B$,

$$f^*(\xi) = \text{trivial} \leftarrow \text{later}$$

Lemma (later): All bundles over B^n (or contractible) a trivial.

$\boxed{\text{Canonical bundles over } \mathbb{RP}^n \quad (n \leq \infty)}$

Recall, $\mathbb{RP}^n = \{\text{1-dim subspaces of } \mathbb{R}^{n+1}\}$ and we have

$$\mathbb{RP}^n \hookrightarrow \mathbb{RP}^{n+1} \hookrightarrow \cdots \hookrightarrow \mathbb{RP}^\infty = \varinjlim \mathbb{RP}^n$$

To each pt $\quad [\ell] \in \mathbb{RP}^n$ get a 1-dim susbpace $\ell \subseteq \mathbb{R}^{n+1}$.

$\uparrow$ line

Define the canonical bundle γ_n^1 over $\mathbb{RP}^n$ as follows:

$$E(\gamma_n^1) := \{([\ell], v) \mid v \in \ell\} \subseteq \mathbb{RP}^n \times \mathbb{R}^{n+1} \text{ and}$$

(subspace topology)

$$\pi([\ell], v) = [\ell]$$

$\gamma_n^1 = E(\gamma_n^1) \xrightarrow{\pi} \mathbb{RP}^n$ is a vector bundle of rank 1.

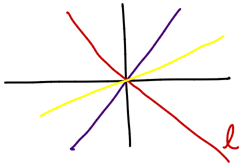
- γ_n^1 is a rank 1 sub-bundle of $\underbrace{\mathbb{RP}^n \times \mathbb{R}^{n+1} \longrightarrow \mathbb{RP}^n}_{\text{product bundle}}$

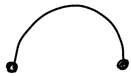
<u>Ex:</u> $n = 1$

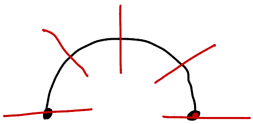
$\mathbb{RP}^1 = \{\text{lines in } \mathbb{R}^2\}$

$= S^1/_{x\sim -x}$

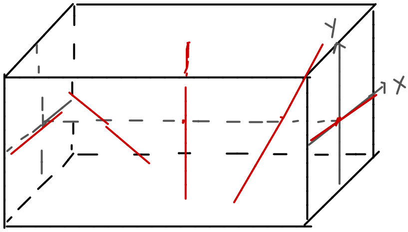
$= I/_{0\sim 1}$







Consider product bundle $\mathbb{RP}^1 \times \mathbb{R}^2 = I \times \mathbb{R}^2/_{(0,\vec{x}) \sim (1,\vec{x})}$



This is exactly the Mobius bundle

$\Rightarrow \quad \gamma_1^1$ = Mobius bundle $\neq$ trivial

<u>HW</u>: Consider $\mathbb{RP}^n \xhookrightarrow{i} \mathbb{RP}^{n+1}$.

Show that $i^*(\gamma_{n+1}^1) = \gamma_n^1$ (easy)

$\uparrow$ Restriction of γ_{n+1}^1 to $\mathbb{RP}^n$.

$\cancel{A} \Rightarrow \gamma_n^1$ is non-trivial for all n!

Can do same for complex v.bs.

$\mathbb{CP}^n = \{$1-dim complex v.s. of $\mathbb{C}^{n+1}\}$.

$$E(\gamma_n^1) = \{([\ell], v) \mid \vec{v} \in \ell\} \subseteq \mathbb{CP}^n \times \mathbb{C}^{n+1}$$

ℓ in a 1-dim $\mathbb{C}$ subspace

$$\downarrow$$

$$\mathbb{CP}^n$$

$$\pi([\ell], v) = [\ell]$$

$\boxed{n=1}$ $\mathbb{CP}^1 = S^2$, $\exists \infty$ many 1-dim $\mathbb{C}$ bundles over S^2

Will classify later.

will see this is non-trivial $\forall n$.