Bundles - Day 13

Classification of v.b.s over Haussdorff paracompact base space B.

Goal: Define a "universal" rank n vector bundle

$$\gamma^n = E_n \longrightarrow G_n \quad (\mathbb{R} \text{ or } \mathbb{C})$$

that is universal in the following sense:

For any other such bundle $\xi := E \xrightarrow{\pi} B$, $\exists$ map (unique up to homotopy) $f: B \to G_n$ s.t. ξ is the pullback $f^*(\gamma^n)$ of γ^n.

Equivalently, there is a bundle map.

$$\begin{tikzcd}
f^*(E_n) \cong E \arrow[r, "f_*"] \arrow[d, "\pi"] & E_n \arrow[d, "\pi_n"] \\
B \arrow[r, "f"] & G_n
\end{tikzcd}$$

G_n is called a classifying space for rank n v.b. and f is the classifying map for ξ.

We will prove:

Theorem:
$$\left\{ \begin{matrix} \text{isom classes of real} \\ \text{rank } n \text{ v.b. over } B \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{homotopy classes} \\ \text{of maps } B \longrightarrow G_n \end{matrix} \right\}$$

$$\shortparallel$$

$$[B, G_n]$$

$$\xi \longmapsto \text{classifying map}$$

$$f^*(\gamma^n) \longleftarrow\!\shortmid \; f: B \longrightarrow G_n.$$

Ex: $G_1 = \mathbb{R}P^\infty$ so

$$\left\{ \begin{array}{c} \text{1-dim } \mathbb{R} \text{ v.bs over} \\ S^k \end{array} \right\} \longleftrightarrow [S^k, \mathbb{R}P^\infty]$$

but $\pi_1(\mathbb{R}P^\infty) = \mathbb{Z}_2$ and $\pi_k(\mathbb{R}P^\infty) = 0$ $k \neq 1$

$\Rightarrow \exists$ a unique $\mathbb{R}$-bundle over S^k with $k \neq 1$.

and there are 2 $\mathbb{R}$-bundles over S^1: $S^1 \times \mathbb{R}$

Mobius bundle

Grassmannians and Universal bundles

Motivation:

If $M^1 \subseteq \mathbb{R}^{k+1}$ is a curve in $\mathbb{R}^{k+1}$ then for each $p \in M^1$

$$\mathbb{R} \cong T_p M^1 \subseteq T_p \mathbb{R}^{k+1} \cong \mathbb{R}^{k+1}$$

we get a line in $\mathbb{R}^{k+1}$, i.e. a point in $\mathbb{R}P^k$.

In addition for each $v \in T_p M^1$, get an element of this subspace. So get a bundle map (i.e. $TM \to M$ is pullback)

$$\begin{tikzcd} TM^1 \arrow[r] \arrow[d] & \gamma_k^1 \arrow[d] \\ M^1 \arrow[r, "f"] & \mathbb{R}P^k \end{tikzcd}$$

More generally, if $M^n \subseteq \mathbb{R}^{n+k}$, for each $p \in M$ get

$$\mathbb{R}^n \cong T_p M \subseteq \mathbb{R}^{n+k}$$

an n-dim subspace of $\mathbb{R}^{n+k}$ (a point in

Grassmann manifold! Each $v \in T_pM$ is a vector in this subspace. (an element of the canonical bundle)

$$\begin{tikzcd} TM \arrow[r] \arrow[d] & \gamma^n_{n+k} \arrow[d] \\ M \arrow[r] & G_n(\mathbb{R}^{n+k}) \end{tikzcd}$$

← universal bundle

<u>Def</u>: The <u>Grassman manifold</u> $G_n(\mathbb{R}^{n+k})$ is the set of n-planes through $\vec{0}$ in $\mathbb{R}^{n+k}$.

<u>Lemma</u>: $G_n(\mathbb{R}^{n+k})$ is a compact (smooth) manifold of dimension nk.

Before proving need:

<u>Def</u>: An <u>n-frame</u> in $\mathbb{R}^{n+k}$ is an n-tuple of linearly independent vectors in $\mathbb{R}^{n+k}$.

<u>Def</u>: The <u>Stiefel manifold</u> $V_n(\mathbb{R}^{n+k})$ is the set of n-frames in $\mathbb{R}^{n+k}$.

Need to put a topology on these sets.

<u>Stiefel manifold</u>:

$$\begin{aligned} V_n(\mathbb{R}^{n+k}) &\cong \{n \times (n+k) \text{ matrices } A \mid A \text{ has full rank}\} \\ &\cong \{n \times (n+k) \text{ matrices } A \mid AA^T \text{ has full rank}\} \\ &\cong f^{-1}(\mathbb{R} \setminus \{0\}) \end{aligned}$$

where $f(A) = \det(AA^T)$

Thus $V_n(\mathbb{R}^{n+k})$ is an open subset of $\mathbb{R}^{n(n+k)}$

$\rightsquigarrow$ use subspace topology.

Grassman: There is a surjective map

$$\begin{tikzcd}
V_n(\mathbb{R}^{n+k}) \arrow[d] & \alpha \arrow[d, mapsto] \\
G_n(\mathbb{R}^{n+k}) & \mathrm{span}(\alpha)
\end{tikzcd}$$

Give $G_n(\mathbb{R}^{n+k})$ the quotient topology (set is open $\Leftrightarrow$ inverse image is open)

Note:

Let $V_n^o(\mathbb{R}^{n+k}) = \{$orthonormal n-frames in $\mathbb{R}^{n+k}\}$ then we have natural maps

$$\begin{tikzcd}
V_n^o(\mathbb{R}^{n+k}) \arrow[dr, "q_0"'] & \subseteq V_n(\mathbb{R}^{n+k}) \arrow[r, "\text{Gram-Schmidt}", "\text{(retract)}"'] \arrow[d, "q"] & V_n^o(\mathbb{R}^{n+k}) \arrow[dl, "q_0"] \\
 & G_n(\mathbb{R}^{n+k}) &
\end{tikzcd}$$

So can topologize $G_n(\mathbb{R}^{n+k})$ as quotient of $V_n^o(\mathbb{R}^{n+k})$. Since $V_n^o(\mathbb{R}^{n+k})$ is a closed a bounded subspace of $\mathbb{R}^{n(n+k)}$, it is compact. Thus $G_n(\mathbb{R}^{n+k})$ is compact!

$$V_n^o(\mathbb{R}^{n+k}) \subseteq S^{n+k-1} \times \dots \times S^{n+k-1}$$

Proof that $G_n(\mathbb{R}^{n+k})$ is a manifold.:

Pick $X_0 \cong \mathbb{R}^n \subseteq \mathbb{R}^{n+k}$. Can write $\mathbb{R}^{n+k} \mathrel{\hat{=}} X_0 \oplus X_0^\perp$

Let $\mathcal{U}_0 \cong \left\{ n\text{-planes } Y \subset \mathbb{R}^{n+k} \,\middle|\, \begin{array}{l} \text{the orthogonal projection} \\ \mathbb{R}^{n+k} \xrightarrow[pr_0]{} X_0 \text{ map } Y \text{ onto } X \end{array} \right\}$

Since $pr_0 : Y \longrightarrow X$ is a linear iso for $Y \in \mathcal{U}$, each pt in Y can be written as unique (v,w) for $v \in X_0$ and $w \in X_0^\perp$. Thus Y is the graph of a linear map from $X_0 \longrightarrow X_0^\perp$. Hence we have map

$$\begin{array}{rl} T : \mathcal{U} \longrightarrow & \mathrm{Hom}(X_0, X_0^\perp) \\ & \quad \mathrel{\rotatebox{90}{$\cong$}} \\ & \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^k) \mathrel{\hat{=}} \mathbb{R}^{nk} \end{array}$$

Claim: T is a homeomorphism

- It is clearly 1-1.
- **Onto:** Pick $\varphi \in \mathrm{Hom}(X_0, X_0^\perp)$, define

$$\begin{array}{rcl} S : \mathrm{Hom}(X_0, X_0^\perp) & \longrightarrow & \mathcal{U} \\ \varphi & \longmapsto & \mathrm{image}(\bar{\varphi}) \end{array}$$

$$\begin{array}{rcl} \bar{\varphi} : X_0 & \longrightarrow & X_0 \oplus X_0^\perp \mathrel{\hat{=}} \mathbb{R}^{n+k} \\ v & \longmapsto & v + \varphi(v) \end{array}$$

Then S, T are continuous and inverses.

Note: Easy to check $G_n(\mathbb{R}^{n+k})$ is Hausdorff.

▨