Bundles - Day 13

Classification of v.b.s over Haussdorff paracompact base space B.

Goal: Define a "universal" rank n weter bundle $Y^n = E_n \longrightarrow G_n$ (R or C)

that is universal in the following sense:

For any other such bundle $z: E \xrightarrow{\pi} B$, \exists map (anique up to homotopy) $f: B \to Gn$ s.t. z is the pullback $f^*(x^n)$ or x^n .

Equivalently, there is a bundle map.

$$f^*(E_n) = \underbrace{f_*}_{B} \xrightarrow{f_*} E_n$$

$$B \xrightarrow{f}_{G_n} G_n$$

Gn is called a clasifying space for rank in v.b. and f is the classifying map for 3.

Ex: $G_1 = \mathbb{R}P^{\infty}$ so $\{S^k, \mathbb{R}P^{\infty}\}$ $\{S^k, \mathbb{R}P^{\infty}\}$ but $T_1(\mathbb{R}P^{\infty}) = \mathbb{Z}_2$ and $T_k(\mathbb{R}P^{\infty}) = 0$ $k \neq 1$ $\Rightarrow \exists \alpha \text{ unique } \mathbb{R}\text{-bundle over } S^k \text{ with } k \neq 1$.

and there are $2 \mathbb{R}\text{-bundle over } S^1 : S^1 \times \mathbb{R}$ Mobius bundle

Grassmannians and Universal bundles

Motivations

If $M' \subseteq \mathbb{R}^{k+1}$ is a curve in \mathbb{R}^{k+1} then for each $p \in M'$ $\mathbb{R} \cong \mathbb{T}_p M' \subseteq \mathbb{T}_p \mathbb{R}^{k+1} \cong \mathbb{R}^{k+1}$

we get a line in 12th, i.e. a point in 12pt.

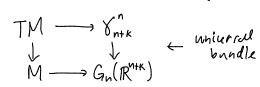
In addition for each VETpM', get an element of this subspace. So get a bundle map (i.e. TM M is pullback)

$$\begin{array}{ccc}
\top M' & \longrightarrow & \chi'_{k} \\
\downarrow & & \downarrow \\
M' & \xrightarrow{f} & \mathcal{P}D^{K}
\end{array}$$

More generally, if $M \subseteq \mathbb{R}^{n+k}$, for each pe M get $\mathbb{R}^n \cong T_{\mathbb{R}}M \subseteq \mathbb{R}^{n+k}$

an n-dim subspace of 1Rn+k (a pint in

Grassmann manifold! Each veTpM is a vector in this subspace. (an element of the canonical bundle)



<u>Def</u>: The <u>Grassman manifold</u> G_n(R^{n+k}) is the set of n-planes through of in R^{n+k}.

Lemma: Gn(Rn+k) is a compact manifold of dimension nk.

Before proving need:

<u>Def</u>: An <u>n-frame</u> in R^{n+k} is an n-tuple of linearly independent vectors in R^{n+k}.

Def: The Stiefel manifold $V_n(\mathbb{R}^{n+k})$ is the set of n frames in \mathbb{R}^{n+k} .

Need to put a topology on these sets.

Stiefel manifold:

$$V_n(\mathbb{R}^{n+k}) \cong \{n \times (n+k) \text{ matrices } A \mid A \text{ has full rank}\}$$

$$\cong \{n \times (n+k) \text{ mutrices } A \mid AA^T \text{ has full rank}\}$$

$$\cong \int_{-1}^{-1} (\mathbb{R} \setminus \{0\})$$
where $f(A) = def(AA^T)$

Thus Vn(Rn+k) is an open subset of Rn(n+k) - use subspace topology.

Grassman: There is a surjective map

$$V_n(\mathbb{R}^{n+k})$$
 \xrightarrow{d} \downarrow $G_n(\mathbb{R}^{n+k})$ span (3)

Give Gn(Rn+k) the quotient topology (set is open (inwrse image is open)

Note:

Let $V_n^o(\mathbb{R}^{n+k}) =$ 3 orthonormal n-frames in \mathbb{R}^{n+k} 3 then we have natural maps

$$V_n^{\circ}(\mathbb{R}^{n+k}) \leq V_n(\mathbb{R}^{n+k}) \xrightarrow{Gran-Schmidt} V_n^{\circ}(\mathbb{R}^{n+k})$$

$$Q_0 \qquad Q_0 \qquad Q_0$$

$$G_n(\mathbb{R}^{n+k})$$

So can topologize Gn(Rn+16) as quotient of Vn(Rn+16). Since Vn(Rn+k) is a closed a bounded subspace of Rn(n+k), it is compact. Thus Gn(Rn+k) is compact! $V_{-}^{\circ}(\mathbb{R}^{n+k}) \leq S^{n+k-1} \times \cdots \times S^{n+k-1}$

$$\sqrt{n}(\mathbb{R}^{n+k}) \leq 5^{n+k-1} \times \cdots \times 5^{n+k-1}$$

Proof that Gn(Rn+k) is a manifold.: Pick Xo= IRn= Rn+k. Can write Rn+k= X, ⊕ X. Since pr.: $Y \longrightarrow X$ is a linear iso for $Y \in U$, each pt in Y can be written as unique (v, w) for VEX, and WEX. Thus Y is the graph of a linear map from Xo -> Xo. Hence we have map $T: U \longrightarrow Hom(X_o, X_o^{\perp})$ Hom (R" 1Rk) = 1Rnk Claim: T is a homeomorphism · It is clearly 1-1. · Onto: Pick 4 & Hom(Xo, Xo), define $S: Hom(X_0, X_1) \longrightarrow U$ 4 → image (4)

Then SIT are continuous and inverses.

Note: Easy to check $G_n(\mathbb{R}^{n+k})$ is Haussdorff.

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 $\varphi: X_0 \longrightarrow X_0 \oplus X^{\perp} \supseteq \mathbb{R}^{N+k}$

 $V \longmapsto V + \varphi(V)$