Day 14

As before, we can define the canonical bundle $\begin{array}{c}\gamma^n_{n+k}\\ \downarrow \\ G_n(\mathbb{R}^{n+k})\end{array}$ as follows:

$$E(\gamma^n_{k+n}) = \left\{ ([\ell], v) \;\middle|\; \begin{array}{l}[\ell] = n\text{-plane in } \mathbb{R}^{n+k}\\ v \in \ell\end{array} \right\} \underbrace{\subseteq G_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k}}_{\text{sub-bundle}}$$

$$E(\gamma^n_{n+k}) \longrightarrow G_n(\mathbb{R}^{n+k})$$
$$([\ell], v) \longmapsto [\ell]$$

Similarly, can define γ^n s.t. $E(\gamma^n) \subseteq G_n(\mathbb{R}^\infty) \times \mathbb{R}^n$.

Will show that γ^n is <u>universal</u> for bundles over paracompact, Haussdorff spaces in the sense that each one is the pullback of γ^n.

$\boxed{\text{Step 1}}$ Suppose B is <u>compact</u> and $\xi = E \xrightarrow{\pi} B$ is a rank n vector bundle, then $\exists\ f : B \to G_n(\mathbb{R}^{n+k})$ s.t. $\xi \cong f^*(\gamma^n_{n+k})$.

Let $U_1, \dots, U_r$ be open sets of B s.t. $\xi|_{U_i}$ is trivial,

$$U_i \times \mathbb{R}^n \xleftarrow{\varphi_i} \pi^{-1}(U_i)$$

Let $h_i = pr_2 \circ \varphi_i$ where $pr_2 : U_i \times \mathbb{R}^n$ is projection onto the second factor. Choose (open) covering $\{V_i\}$ s.t. $\overline{V_i} \subseteq U_i$ and $W_i \subseteq V_i$ s.t. $\overline{W_i} \subseteq V_i$. Take bump functions $\lambda_i : B \to \mathbb{R}$ s.t.

$$\lambda_i = \begin{cases} 1 & \text{on } \overline{W_i} \\ 0 & \text{on } B \setminus V_i \\ \text{between } 0 \text{ and } 1 \text{ everywhere else} \end{cases}$$

Let $h_i' : E \longrightarrow \mathbb{R}^n$ be defined by

$$h_i'(e) = \begin{cases} 0 & \text{if } \pi(e) \notin \bar{V}_i \\ \lambda_i(\pi(e)) \cdot h_i(e) & \text{if } \pi(e) \in U_i \end{cases}$$

<u>Note</u>: This is a linear map on a fixed E_b since it is either 0 or a constant, $\lambda_i(b)$, times h_i. However, it is often not injective!

Define $\bar{f} : E \longrightarrow \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{r}$ by

$$\bar{f}(e) = (h_1'(e), \dots, h_r'(e)).$$

this is continuous, linear on each E_b plus for each b, $\exists\ i$ s.t. $\lambda_i(b) = 1 \Rightarrow h_i'(e) = h_i(e)$ for $e \mapsto b$. Hence $E \xrightarrow{h_i'} \mathbb{R}^n$ is an isomorphism. $\Rightarrow \bar{f}$ is an isom.

Thus, we get a bundle map as follows:

$$\begin{array}{ccc} E & \xrightarrow{F} & E(\gamma^n_{nr}) \subseteq G_n(\mathbb{R}^{nr}) \times \mathbb{R}^{nr} \\ \pi \downarrow & & \downarrow \\ B & \xrightarrow{f} & G_n(\mathbb{R}^{nr}) \end{array}$$

$\boxed{\mathbb{R}^{nr} = \mathbb{R}^{n+k} \text{ with } k = (r-1)n}$

- $F(e) := (\underbrace{[\bar{f}(E_{\pi(e)})]}_{\text{n-plane}}, \bar{f}(e))$
- $f(b) := [\bar{f}(E_b)]$

$\boxed{\text{Step 2}}$ B is not compact (but paracompact)

Need to use infinite Grassmann in this case!

Theorem: Any rank n vector bundle over a paracompact Haussdorff B admits a bundle map $\xi \to \gamma^n$.

- To prove this, we mimic the compact case using a countably infinite but locally finite cover $\{U_i\}$ s.t. $\xi|_{U_i}$ is trivial. Proceed as before. $\blacksquare$