$\boxed{\text{Day 15}}$ We are proving the classification theorem

<u>Theorem</u>: For B paracompact and Haussdorff,

$$\left\{\begin{matrix}\text{homotopy} \\ \text{classes of maps} \\ B \to G_n(\mathbb{R}^\infty)\end{matrix}\right\} =: [B, G_n(\mathbb{R}^n)] \xrightarrow[\cong]{\varphi} \mathrm{Vect}^n(B) = \left\{\begin{matrix}\text{rank } n \text{ vector} \\ \text{bundles over } B\end{matrix}\right\}$$
$$[f] \longmapsto f^*(\gamma^n)$$

Last time, we showed that φ is surjective.

$\boxed{\varphi \text{ injective}}$

Recall from last lecture that if one has a map $\hat{F}: E \to \mathbb{R}^\infty$ that is linear and injective on each fiber $E_b \xrightarrow{\hat{F}} \mathbb{R}^\infty$, then there is a bundle map

$$\begin{array}{ccc} E & \xrightarrow{F} & E(\gamma^n) \subseteq G_n \times \mathbb{R}^\infty \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & G_n \end{array} \qquad (G_n = G_n(\mathbb{R}^\infty))$$

where $F(e) := ([\hat{F}(E_{\pi(e)})], \hat{F}(e))$ $f(b) = ([\hat{F}(E_b)])$.

Conversely, if one has a bundle map

$$\begin{array}{ccc} E & \xrightarrow{F} & E(\gamma^n) \subseteq G_n(\mathbb{R}^\infty) \times \mathbb{R}^\infty \\ \pi\downarrow & & \downarrow \\ B & \xrightarrow{f} & G_n(\mathbb{R}^\infty) \end{array}$$

Then $F: E_b \to \{f(b)\} \times \mathbb{R}^\infty$ is linear and injective.

Let $\hat{F}: E \to \mathbb{R}^\infty$ be $\hat{F} = pr_2 \circ F$ where pr_2 = projection to 2nd factor. So $\hat{F}: E_b \to \mathbb{R}^\infty$ is linear and injective.

Thus bundle maps to γ^n are equivalent to maps

$$F: E \to \mathbb{R}^\infty$$

that are linear and injective on each fiber.

Theorem 1: Any two bundle maps from a rank n vector bundle ξ to γ^n are bundle-homotopic

Proof: Given bundle maps, get $\hat{F}_i : E \longrightarrow \mathbb{R}^\infty$ linear and injective on each fiber. It suffices to produce a homotopy $\hat{F}_t : E \longrightarrow \mathbb{R}^\infty$ that is linear and injective for all t on each fiber.

There are two cases to consider

$\boxed{\text{Case 1:}}$ $\hat{F}_0(e)$ is never a multiple of $\hat{F}_1(e)$ whenever $e \neq \vec{0}$.

In this case, we can define

$$\hat{F}_t(e) = (1-t)\hat{F}_0(e) + t\hat{F}_1(e)$$

Then $\hat{F}_t$ is lin. and injective on each fiber.

$\boxed{\text{Case 2:}}$ Define

$$d_{even} : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty \qquad \text{and} \qquad d_{odd} : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$$
$$e_i \longmapsto e_{2i} \qquad\qquad\qquad e_i \longmapsto e_{2i+1}$$

to be the linear injective maps where $e_i = (0, \dots, 0, \underset{\substack{\uparrow \\ i^{th}}}{1}, 0, \dots)$.

Then $\hat{F}_0$ and $d_{odd} \circ \hat{F}_0$ so we can use case 1 to show they are homotopic with the homotopy linear and injective on each fiber. Do the same for $\hat{F}_1$ and $d_{even} \circ \hat{F}_1$. Moreover, the images of $d_{odd} \circ \hat{F}_0$ and $d_{even} \circ \hat{F}_1$ only intersect in $\vec{0}$; thus they satisfy Case 1 as well. Thus

$$\hat{F}_0 \sim d_{odd} \circ \hat{F}_0 \sim d_{even} \circ \hat{F}_1 \sim \hat{F}_1 .$$

$\blacksquare$

Suppose $\Psi([f]) = \Psi([g])$, that is $f, g : B \longrightarrow G_n(\mathbb{R}^\infty)$ with $f^*(\gamma^n) \cong g^*(\gamma^n)$. Then we have bundle maps

$$\begin{tikzcd} f^*(\gamma^n) \arrow[r, "\tilde{f}"] \arrow[d] & \gamma^n \arrow[d] \\ B \arrow[r, "f"] & G_n(\mathbb{R}^\infty) \end{tikzcd} \quad + \quad \begin{tikzcd} f^*(\gamma^n) \arrow[r, "\eta"] \arrow[dr] & g^*(\gamma^n) \arrow[r, "\hat{g}"] \arrow[d] & \gamma^n \arrow[d] \\ & B \arrow[r, "g"] & G_n(\mathbb{R}^\infty) \end{tikzcd}$$

By the previous theorem, the bundle maps $\tilde{f}$ and $\hat{g} \circ \eta$ are bundle homotopic, and so the maps the cover are as well. Hence $f \sim g$. Thus Ψ is injective!

We have shown everything except well-definedness.

$\boxed{\Psi \text{ is well-defined}}$

<u>Theorem 1</u>: If $\xi = E \xrightarrow{\pi} B$ is a vector bundle over B $f_0, f_1 : X \longrightarrow B$ are homotopic maps from X a paracompact and Haussdorff space, then $f_0^*(\xi)$ is isomorphic to $f_1^*(\xi)$.

<u>Note</u>: This is also true for principal G-bundles.

We will prove a slightly stronger theorem.

<u>Theorem 1'</u>: If $E \xrightarrow{\pi} X \times [0,1]$ is a vector bundle with X paracompact and Haussdorff then $E \cong E|_{X \times \{1\}} \times [0,1]$ where

$$\begin{tikzcd} E|_{X\times\{1\}} \times [0,1] \arrow[d] & (e,t) \arrow[d, mapsto] \\ X \times [0,1] & \pi(e) \end{tikzcd}$$

Hence $E|_{X \times \{0\}} \cong E|_{X \times \{1\}}$ (as bundles).

$\boxed{\text{Theorem } 1' \Rightarrow \text{Theorem } 1}$: Suppose $f_0 \sim f_1$ via $F: X \times I \to B$. Then $F^*(\xi)$ is a bundle over $X \times [0,1]$, and its restriction to $X \times \{i\}$ is isomorphic to $f_i^*(\xi)$ for $i = 0, 1$:

$$\begin{tikzcd}
f_i^*(\xi) = (F \circ j_i)^*(\xi) \arrow[r] \arrow[d] & F^*(\xi) \arrow[r] \arrow[d] & E \arrow[d, "\pi"', "= \xi"] \\
X \times \{i\} \arrow[r, "j_i"] \arrow[rr, bend right, "f_i"'] & X \times [0,1] \arrow[r, "F"] & B
\end{tikzcd}$$

Use naturality of pullbacks:

$$(g \circ f)^*(\xi) \cong g^*(f^*(\xi))$$

So by Theorem $1'$, $f_0^*(\xi) \cong f_1^*(\xi)$.

- We will prove Theorem $1'$ next time!

$\boxed{\text{Important Corollary}}$: Any bundle over a contractible, paracompact, Haussdorff base space is trivial.

Proof: Since B is contractible, $id: B \to B$ is homotopic to a constant map $c: B \to B$. Theorem 1 implies

$$\xi \cong (id)^*(\xi) \cong c^*(\xi) \cong \text{product bundle}.$$