$\boxed{\text{Day 16}}$

Lemma: Suppose $\xi : E \xrightarrow{\pi} X$ where $X = A \cup B$ where $\xi|_A$ and $\xi|_B$ are trivial and $A \cap B$ is a retract of B. Then ξ is trivial

Cor: A bundle whose base space is a closed n-ball is trivial

Proof of Lemma: Choose trivializations of $\xi|_A$, ξ_B

$$A \times \mathbb{R}^n \xrightarrow{\phi_A} \pi^{-1}(A) \qquad \pi^{-1}(B) \xleftarrow{\phi_B} B \times \mathbb{R}^n$$

Let $C = A \cap B$

$$\begin{aligned} C \times \mathbb{R}^n &\xrightarrow{\phi_A} \pi^{-1}(C) \xrightarrow{\phi_B^{-1}} C \times \mathbb{R}^n \\ (c, v) &\longmapsto (c, g(c) v) \qquad g : C \longrightarrow GL_n(\mathbb{R}) \end{aligned}$$

If $g_c = id$, we can glue the two together to get trivial v.b.

We alter ϕ_B (new trivialization of $\xi|_B$):

$$B \times \mathbb{R}^n \xrightarrow{R?} B \times \mathbb{R}^n \xrightarrow{\phi_B} \pi^{-1}(B)$$

Since C is a retract of B, we have
$$\begin{tikzcd} C \arrow[r, "i"] \arrow[rr, bend right, "id"'] & B \arrow[r, "r"] & C \end{tikzcd}$$
and we thus can extend g to B:

$$\begin{aligned} \tilde{g} : B &\longrightarrow GL_n(\mathbb{R}) \\ b &\longmapsto g_c(r(b)) \end{aligned}$$

and so can let
$$\begin{aligned} R : B \times \mathbb{R}^n &\longrightarrow B \times \mathbb{R}^n \\ (b, v) &\longmapsto (b, \tilde{g}(b) v) \end{aligned}$$

Consider new trivialization $\phi_B' : B \times \mathbb{R}^n \longrightarrow \pi^{-1}(B)$ by

$$\phi_B' = \phi_B \circ R$$

Then on C, we have

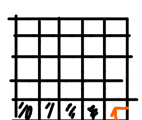
$$C \times \mathbb{R}^n \xrightarrow{\phi_A} \pi^{-1}(C) \xrightarrow{(\phi_B')^{-1} = \tilde{R}^{-1} \circ \phi_B^{-1}} C \times \mathbb{R}^n$$

$$(c, v) \longmapsto (c, \underbrace{(\tilde{g}(c)^{-1} \circ g(c))}_{= id} v)$$

$$= (c, v)$$

Since ϕ_A agrees with ϕ_B on all of C, we can paste the two trivializations together to get a trivial bundle on all of $A \cup B$.

<u>Proof of Corollary</u>: Consider $\xi = E \xrightarrow{\pi} B^n$. Fix an atlas $\{U_i\}$ for ξ (i.e. $\xi|_{U_i}$ is trivial). Decompose $B^n = I \times \cdots \times I$ into small cubes B_i. Since B^n is a compact metric space, $\exists$ Lebesgue # δ s.t. if $\operatorname{diam}(B_i) < \delta$, then each cube is contained in some U_i so $\xi|_{B_i}$ is trivial. There are a finite # of cubes, and can build B^n add one cube at a time s.t. the intersection is a face of the cube added (so is a retract).


(add each B_i one at a time)

To generalize this, need to recall paracompactness.

<u>Def</u>: A space X is paracompact if every open covering $\mathcal{U}$ of X has a locally finite open refinement $\mathcal{V}$ that covers X.

- for $V \in \mathcal{V}$, $\exists$ $U \in \mathcal{U}$ s.t. $V \subset U$.
- $\forall$ $x \in X$, $\exists$ nbhd W that intersects only finite # of V's in $\mathcal{V}$