Day 17

Last time we defined paracompact.

Note: • every metrizable space is paracompact.
• every manifold is paracompact.

Theorem (see Munkres Thm 41.7): Let X be a paracompact, Haussdorff space. Then there is a partition of unity on X dominates by $\{U_\alpha\}$. i.e. $\exists$ continuous $\lambda_\alpha : X \longrightarrow [0,1)$ s.t

(1) support $\lambda_\alpha \subseteq U_\alpha$

(2) $\{\text{support } \lambda_\alpha\}_\alpha$ is locally finite

(3) $\Sigma \lambda_\alpha(x) = 1 \quad \forall \; x \in X$.

If X is a smooth mfld, then $\exists$ smooth partition of unity.

Note: partions of unity are useful for patching local functions together to get a global function.

Ex: If $f_\alpha : U_\alpha \longrightarrow \mathbb{R}^n$ smooth $\Rightarrow f = \Sigma' \lambda_\alpha f_\alpha$ is smooth and well-defined since

• $\lambda_\alpha f_\alpha$ smooth on U_α
• $\lambda_\alpha f_\alpha$ smooth on $X \setminus \text{supp}\, \phi_\alpha$ since it is defined to be zero.
• For any $x \in X$, x only lives in a finite # of U_α so $\Sigma' \lambda_\alpha f_\alpha(x)$ finite.
• f is smooth $\Leftrightarrow$ it is smooth on each open subset of X

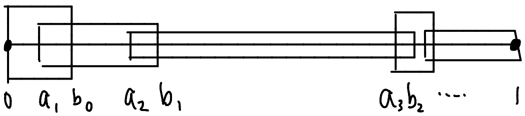
For each x, $\exists$ open V_x s.t. $V_x \cap U_\alpha = \emptyset$ except finite # α's. Hence on V_α, $\Sigma' \lambda_\alpha f_\alpha$ is a finite # of smooth functions.

Theorem 2': If $\xi = E \xrightarrow{\pi} X \times [0,1]$ is a vector bundle with X paracompact and Hausrdorff then

$$E|_{X\times\{0\}} \cong E|_{X\times\{1\}} \quad \text{(as bundles)}.$$

Proof:

$\boxed{\text{Step 1}}$ Want to find an open cover of X, $\{U_\alpha\}$ s.t. $\xi|_{U_\alpha \times [0,1]}$ is trivial. Fix $x \in X$ and consider an open cover of $\{x\}\times[0,1]$ by charts over which the bundle is trivial. Can assume (by product topology) that open sets are form $U_{x,i} \times (a_i, b_i)$ (or $U_{x,0} \times [0,b_0)$, $U_{x,m} \times (a_m, 1]$)



Note: $\exists$ a "tube" inside all of them.

Since $[0,1]$ is compact, can use a finite # of them.
Let $U_x := \bigcap_{i=1}^{m} U_{x,i}$, can use previous lemma (with $(a_i, b_i) \cap (a_{i+1}, b_{i+1})$ a deformation retract of (a_{i+1}, b_{i+1})) to show that $\xi|_{U_x \times [0,1]}$ is trivial.

$\boxed{\text{Step 2}}$ Find a partition of unity subordinate to $\{U_x\}_{x\in X}$.
For simplicity, assume X = compact (see MS for entire proof).
Then there is a finite subcover and a finite partion of

unity. $\lambda_1, \dots, \lambda_r$. Consider the graph of $\lambda_1 + \cdots + \lambda_r$, i.e. $gr_r = \{ (x, \lambda_1(x) + \cdots + \lambda_r(x)) \} \subseteq X \times [0,1]$.

Since $\lambda_1(x) + \cdots + \lambda_r(x) = 1$, we have that $gr_r = X \times \{1\}$.

Now consider the bundle over

$$gr_{r-1} = \{ (x, \lambda_1(x) + \cdots + \lambda_{r-1}(x)) \mid x \in X \} \subseteq X \times [0,1]$$

Note: $\mathrm{supp}(\lambda_r) \subseteq U_r \times [0,1]$ and $\xi|_{U_r \times [0,1]} =$ trivial.

So for $x \notin U_r$, $\quad \lambda_1(x) + \cdots + \lambda_r(x) = \lambda_1(x) + \cdots + \lambda_r(x)$

$$\Rightarrow gr_{r-1} \cap U_i \times [0,1] = gr_r \cap U_i \times [0,1] \text{ for } i \neq r.$$

Thus gr_{r-1} and gr_r only differ on $U_r \times [0,1]$.

But these are both trivialized by single chart on $U_r \times [0,1]$.
and their intersection is trivialized by this chart,
so they are isomorphic.

Keep doing this until you get to $gr_0 = \{ (x, 0) \mid x \in X \}$.

Then $\xi|_{X \times \{0\}} \cong \xi|$.

Universal bundles for other situations (B = paracompact, Haus.)

- For complex vector bundles, we have $\gamma^n \longrightarrow G_n(\mathbb{C}^\infty)$ complex Grassmann
  and $\gamma^n =$ canonical n-plane bundle over G_n.

$$\left\{ \begin{array}{l} \text{homotopy classes} \\ \text{of map to } G_n(\mathbb{C}^\infty) \end{array} \right\} \longleftrightarrow \mathrm{Vect}^n_{\mathbb{C}}(B)$$

- For principal G-bundles (G = top group) or vector bundles with restricted structure groups, $SO(n)$, $Spin(n)$, etc., have universal bundles $\begin{matrix} EG \\ \downarrow \\ BG \end{matrix}$ and bundles are classified by classifying maps to BG.