Day 18

Applications of the classification theorem:

Real line bundles over B:

$$\mathrm{Vect}^1(X) = [X, \mathbb{R}P^\infty] = [X, K(\mathbb{Z}_2, 1)] = H^1(X; \mathbb{Z}_2)$$

$\Rightarrow X = S^m$ we have

$$H^1(S^m; \mathbb{Z}_2) = \begin{cases} 0 & m \neq 1 \\ \mathbb{Z}_2 & m = 1 \end{cases}$$

Only trivial bundle when $m \neq 1$, get product + Mobius bundle over S^1.

Complex line bundles over X: (X = CW complex or manifold)

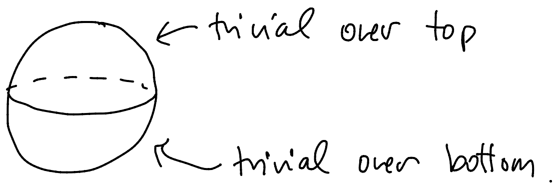
$$\mathrm{Vect}^1_{\mathbb{C}}(X) = [X, G_1(\mathbb{C}^\infty)] = [X, \mathbb{C}P^\infty] = [X, K(\mathbb{Z}, 2)] \underset{\text{alg. top.}}{\cong} H^2(X; \mathbb{Z})$$

$\Rightarrow$ when $X = S^m$, $H^2(S^m; \mathbb{Z}) = 0$ for $m \neq 2$.

$$H^2(S^2; \mathbb{Z}) \cong \mathbb{Z}$$

$\Rightarrow$ there are $\mathbb{Z}$ worth of complex line bundles over S^2. Can see this will clutching construction,

Consider a bundle over S^2:



These attach along equator S^1.

$$\begin{aligned} S^1 \times \mathbb{C} &\longrightarrow S^1 \times \mathbb{C} \\ (x, \vec{z}) &\longmapsto (x, g(x)\vec{z}) \end{aligned} \qquad \underset{\text{transition map.}}{g : S^1 \longrightarrow GL_1(\mathbb{C}) \cong U(1)}$$

It turns out that the bundle only depends on homotopy class of g. $[g] \in \pi_1(U(1)) \cong \mathbb{Z}$.

Note: structure group is always in $SO(2)$ so is orientable!

<u>Clutching construction</u> $\quad \xi = E \xrightarrow{\pi} S^n$

Let ξ be a fiber bundle w/ fiber F over S^n and structure gp $G \subseteq \mathrm{Homeo}(F,F)$. Let N = northern hemisphere and S = southern hemisphere. Then $\xi|_N$ and $\xi|_S$ are trivial.

$$\pi^{-1}(N) \xleftarrow{\phi_N} N \times F$$

$$\pi^{-1}(S) \xleftarrow{\phi_S} S \times F$$

Over $S^{n-1} = N \cap S$

$$\begin{aligned} S^{n-1} \times F \xrightarrow{\phi_N} \pi^{-1}(S^{n-1}) &\xrightarrow{\phi_S^{-1}} S^{n-1} \times F \\ (x, v) &\longmapsto (x, f(x)v) \end{aligned} \qquad f : S^{n-1} \longrightarrow G$$

Note: All vector bundles over S^n for $n \geq 2$ are orientable since transition function like in a connected component of $GL_n(\mathbb{R})$, i.e. can reduce structure group to $SO(n)$!

Moreover, given $f: S^{n-1} \longrightarrow G$, can construct a bundle ξ_f by gluing product bundles together.

Thm: For G path connected or $n=1$, $\exists$ bijection

$$\pi_{n-1}(G) \underset{\Psi}{\overset{\cong}{\longleftrightarrow}} \left\{ \begin{array}{l} \text{isomorphism classes} \\ \text{of } F \text{ bundles over } S^n \\ \text{with structure gp } G \end{array} \right\}$$

$$[f] \longmapsto \xi_f.$$

Proof: Step 1 is to show that Ψ is well-defined.

Suppose $f_0 \sim f_1$ via homotopy F_t. Let

$$\xi_i = \xi_{f_i} = \begin{array}{c} E_i \\ \downarrow \pi_i \\ S^n \end{array}.$$

Define iso $\xi_0 \xrightarrow{\eta} \xi_1$. Let $\eta: E_0|_N \longrightarrow E_1|_N$ be identity.

$$\xi_f = S \times F \cup_f N \times F$$
$$(x,v) \sim (x, f(x)v)$$

So on $S^{n-1} \times F \subset S \times F$ we have

$$(*)\quad \begin{array}{ccccccc} S\times F & & N\times F & & N\times F & & S\times F \\ \cup & & \cup & & \cup & & \cup \\ S^{n-1}\times F & \xrightarrow[\text{in } \xi_0]{\text{gluing}} & S^{n-1}\times F & \xrightarrow{id} & S^{n-1}\times F & \xrightarrow[\text{in } \xi_1]{\text{gluing}} & S^{n-1}\times F \\ (x,v) & \longmapsto & (x, f_0(x)^{-1}v) & \longrightarrow & (x, f_0(x)^{-1}v) & \longrightarrow & (x, f_1(x) f_0(x)^{-1} v) \end{array}$$

Want to extend this to all of $S\times F$.

Note that G is a top group so the map

$$f_0^{-1} : S^{n-1} \longrightarrow G \quad \text{by} \quad f_0^{-1}(x) := f_0(x)^{-1}$$

makes sense and is continuous. Also $f_1 \circ f_0^{-1} : S^{n-1} \longrightarrow G$ makes sense, and is continuous. Can check that since $f_0 \sim f_1$, $f_1 \circ f_0^{-1}$ is homotopic to constant map, c_e, sending each element to $e \in G$. To prove this, note $F_t \circ f_0^{-1}$ is such a homotopy.

Thus $f_1 \circ f_0^{-1}$ extends to $\bar{f} : S \longrightarrow G$, hence can extend $(*)$ to $S\times F \longrightarrow S\times F$, which gives us our isomorphism.

$$(x,v) \longmapsto (x, \bar{f}(x) v)$$

$\boxed{\Psi \text{ injective}}$ Suppose $\xi_f \cong \xi_g$ for some f, g.

We have

$$\begin{array}{ccccc} (x,v) & N\times F & \xrightarrow{\eta_N} & N\times F & (x,v) \\ \downarrow & \cup & & \cup & \\ & S\times F & \xrightarrow{\eta_S} & S\times F & \\ (x, f(x) v) & & & & (x, g(x), v) \end{array}$$

By composing with another iso can assume $\eta_S = id$.

$$
\begin{tikzcd}
(x,v) \arrow[r] \arrow[d] & \overset{\eta_N(x,v)}{\overset{\shortparallel}{(x, g^{-1}(x)f(x)v)}} \\
(x,f(x)v) \arrow[r] & (x,f(x)v) \arrow[u]
\end{tikzcd}
$$

Hence the map $g^{-1}\circ f$ extends to N via η_N.

$$\eta_N(x,v) = (x, \bar{\eta}(x)v)$$

So $g^{-1}\circ f \simeq$ constant, via $F_t : S^{n-1} \longrightarrow G$.

Let $H_t : S^{n-1} \longrightarrow G$ be defined by $H_t = g\circ F_t$.

Then $f \sim g$.

$\boxed{\Psi \text{ surjective}}$ Already proved.

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