$\boxed{\text{Day 19}}$ Characteristic classes

Fix a category of bundles for which $EG \xrightarrow{\pi} BG$ is the universal bundle. For examples, for real rank n vector bundles, we have $\gamma^n = E(\gamma^n) \xrightarrow{\pi} G_n(\mathbb{R}^\infty)$.

For any abelian group A and any class $c \in H^i(BG; A)$, we can assign to any bundle ξ, the characteristic class $c(\xi) := f^*(c) \in H^i(B; A)$ where $f: B \longrightarrow BG$ is a classifying map for ξ.

Proposition: $c(\xi)$ is well-defined and independent of choice of f. Moreover, if $\xi \cong \xi'$ then $c(\xi) = c(\xi')$. Hence if ξ = trivial, then $c(\xi) = 0$ for any characteristic class c.

Proof: If f, g are classifying maps for ξ and ξ' $f \sim g$. Hence $f^* = g^*$ on $H^i(-; A)$, so $f^*(c) = g^*(c)$. If $\xi \cong B \times F$ then $\xi = f^*(EG \to BG)$ where f is a constant map. But then $f^* = 0$ on all cohomology groups. $\blacksquare$

Behavior under bundles maps

Proposition:

$$\text{If} \quad \xi' = \begin{array}{ccc} E' & \xrightarrow{\tilde{g}} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{g} & B \end{array} = \xi \quad \text{is a bundle map,}$$

then $c(\xi') = g^*(c(\xi))$. Hence

$$\boxed{c(g^*(\xi)) = g^*(c(\xi))}$$

<u>Proof</u>: Let $f: B \to BG$ be a classifying map for ξ.

$$\begin{tikzcd}
E' \arrow[r, "\tilde{g}"] \arrow[d] & E \arrow[r, "\tilde{f}"] \arrow[d, "\pi"] & EG \arrow[d] \\
B' \arrow[r, "g"] & B \arrow[r, "f"] & BG
\end{tikzcd}$$

Then $f \circ g$ is a classifying map for ξ' since

$$(g \circ f)^*(EG \to BG) = g^*(f^*(EG \to BG))$$

and $f^*(EG \to BG) \cong \xi'$. Hence

$$c(E') = (g \circ f)^*(c) = g^*(f^*(c)) = g^*(c(E)).$$

Since $E' \cong f^*(EG \to BG)$, the last statement follows. ∎

<u>Questions</u>:

1) How many characteristic classes are there?
   $\to$ Need to figure out cohomology groups of $G_n(\mathbb{R}^\infty)$ and $G_n(\mathbb{C}^\infty)$.

2) If ξ = trivial $\Rightarrow$ all characteristic classes are zero. Is the converse true? i.e. if classifying map induces zero on all cohomology, is it null-homotopic? (dim > 0)

Remark: Need to only consider generators of $H^i(BG;A)$ (or cohomology ring if $A=$ ring):

- if $c_1, c_2 \in H^i(BG;A)$ then
$$(c_1+c_2)(\xi) = f^*(c_1+c_2) = f^*(c_1)+f^*(c_2) = c_1(\xi)+c_2(\xi)$$
- if A is a ring and $c_1, c_2 \in H^*(BG;A)$ then
$$(c_1 \cup c_2)(\xi) = f^*(c_1 \cup c_2) = f^*(c_1) \cup f^*(c_2) = c_1(\xi) \cup c_2(\xi)$$

Stiefel-Whitney classes (for real vector bundles)

Recall $H^i(\mathbb{RP}^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1]$ $\quad w_1 \in H^1(\mathbb{RP}^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2$.

so for all rank 1 vector bundle, we can define
$$w_1(\xi) := f^*(w_1)$$
to be the first Stiefel-Whitney class.

Ex: Mobius bundle over S^1: $E_{MB} \to S^1$

$$\begin{array}{ccc} E_{MB} & \longrightarrow & E(\gamma^1) \\ \downarrow & & \downarrow \\ \mathbb{RP}^1 = S^1 & \overset{i}{\hookrightarrow} & \mathbb{RP}^\infty \end{array}$$

i^* is an $\cong$ on $H^1(-;\mathbb{Z}_2) \cong \mathbb{Z}_2$ so
$$w_1(E_{MB} \to S^1) = i^*(w_1) \neq 0 \text{ in } H^1(S^1;\mathbb{Z}_2) \cong \mathbb{Z}_2$$

$\Rightarrow w_1$ is non-trivial for Mobius bundle.

Theorem: $H^*(G_n(\mathbb{R}^\infty);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \ldots, w_n]$ where $w_i \in H^i(G_n(\mathbb{R}^\infty);\mathbb{Z}_2)$.

That is, every element in $H^p(G_n(\mathbb{R}^\infty);\mathbb{Z}_2)$ can be uniquely written as

$$\sum_{\text{finite}} w_1^{k_1}\cdots w_n^{k_n}$$

where $k_i \geq 0$ $\quad k_1+\cdots+k_n = p$ (mult. corresponds to cup product).

- Will prove this by finding a cell structure for $G_n(\mathbb{R}^\infty)$.

<u>Note</u>: We really should be saying

$$w_{i,n} \qquad 1 \leq i \leq n$$

to specify we are in $H^i(G_n(\mathbb{R}^\infty);\mathbb{Z}_2)$.

We have the following fact

$$\mathbb{R}^\infty \hookrightarrow \mathbb{R}^\infty \times \mathbb{R}$$

sending

$$G_n(\mathbb{R}^\infty) \xhookrightarrow{i_n} G_{n+1}(\mathbb{R}^\infty)$$
$$P \longmapsto P \times \mathbb{R}$$

This induces $\quad (i_n)^*: H^j(G_{n+1}(\mathbb{R}^\infty);\mathbb{Z}_2) \longrightarrow H^j(G_n(\mathbb{R}^\infty);\mathbb{Z}_2)$

and

$$\boxed{(i_n)^*(w_{j,n+1}) = w_{j,n}\,.}$$

However, i_n^* has kernel : $(i_n^*)(w_{n+1,n}) = 0$

Stiefel-Whitney classes in general:

<u>Def</u>: If ξ is a rank n vector bundle over B and $f: B \to G_n(\mathbb{R}^\infty)$ is a classifying map for ξ, define the ith Stiefel-Whitney class of ξ to be

$$\boxed{w_i(\xi) := f^*(w_{i,n}) \in H^i(B;\mathbb{Z}_2)}$$