Day 20

Properties of Stiefel-Whitney classes

Let $\xi : E \xrightarrow{\pi} B$ be a rank n vector bundle, B path conn.

* (0) If $\xi \cong \xi'$ then $w_i(\xi) = w_i(\xi)$

(1) $w_0(\xi) = 1 \in H^0(B;\mathbb{Z}_2) \cong \mathbb{Z}_2$ (by definition)

(2) $w_i(\xi) = 0$ for $i > n$ (by definition)

* (3) If $\xi' =$
$$\begin{tikzcd} E' \arrow[r, "\hat{g}"] \arrow[d] & E \arrow[d, "\pi"] \\ B' \arrow[r, "g"] & B \end{tikzcd}$$
is a bundle map then

then $w_i(\xi') = g^*(w_i(\xi))$.

* (4) If $g : B' \to B$ is a cont. map then
$$w_i(g^*(\xi)) = g^*(w_i(\xi)).$$

* (5) Non-triviality: $w_1(\gamma_1^1) = w_1(\text{Möbius bundle}) = 1$

* (6) $w_i(\varepsilon^n) = 0$ for $i \neq 0$.

(7) w_i is a <u>stable</u> characteristic class, i.e.
$$w_i(\xi \oplus \varepsilon) = w_i(\xi)$$

<u>Note</u>: Technically should say $w_{i,n+1}(\underset{\substack{\uparrow \\ \text{rank } n+1}}{\xi \oplus \varepsilon}) = w_{i,n}(\underset{\substack{\uparrow \\ \text{rank } n.}}{\xi})$

<u>Def</u>: The total Stiefel-Whitney class:
$$w(\xi) := 1 + w_1(\xi) + w_2(\xi) + \cdots \qquad \in H^*(B;\mathbb{Z}_2)$$

(8) If ξ, ξ' are bundles over B, then
$$w(\xi \oplus \xi') = w(\xi)\, w(\xi') \qquad (\text{mult. is } \cup)$$

Hence

$$W_k(\xi \oplus \xi') = \sum_{i=0}^{k} W_i(\xi) W_{k-i}(\xi)$$

<u>Def</u>: $H^{\Pi}(B;\mathbb{Z}_2)$ is the ring of all formal infinite series $a = a_0 + a_1 + a_2 + \cdots$ with $a_i \in H^i(B;\mathbb{Z}_2)$. Here

$$a \cdot b = (a_0 + a_1 + \cdots)(b_0 + b_1 + \cdots)$$
$$:= \underset{H^0}{\underset{\uparrow}{a_0 b_0}} + \underset{H^1}{\underset{\uparrow}{(a_0 b_1 + a_1 b_0)}} + \underset{H^2}{\underset{\uparrow}{(a_1 b_1 + a_0 b_2 + a_2 b_0)}} + \cdots$$

The group of units in $H^{\Pi}(B;\mathbb{Z}_2)$ is the set of elements a with $a_0 = 1$.

What is the inverse of $a = 1 + a_1 + \cdots$?

$$a\bar{a} = 1$$

$$\Rightarrow (1 + a_1 + a_2 + \cdots)(1 + \bar{a}_1 + \bar{a}_2 + \cdots) = 1$$

implies

$$a_1 + \bar{a}_1 = 0 \Rightarrow \bar{a}_1 = a_1$$
$$a_1\bar{a}_1 + a_2 + \bar{a}_2 = 0 \Rightarrow \bar{a}_2 = a_2 + a_1^2$$
$$a_3 + \bar{a}_3 + a_1\bar{a}_2 + a_2\bar{a}_1 = 0 \Rightarrow \bar{a}_3 = a_3 + a_1(a_2 + a_1^2) + a_2 a_1$$
$$= a_3 + a_1^3$$
$$\bar{a}_4 = a_1^4 + a_1^2 a_2 + a_2^2 + a_4$$

In particular, $\bar{a}_n$ can be constructed inductively:

$$\bar{a}_n = a_1\bar{a}_{n-1} + a_2\bar{a}_{n-2} + \cdots + a_n.$$

In addition, since

$$1 = (1+b)(1 \underset{\text{mod } 2}{\underset{\uparrow}{-}} b + b^2 - b^3 + \cdots)$$

$$\bar{a} = (1 + (a_1 + a_2 + \cdots))^{-1}$$
$$= (1 + (a_1 + a_2 + \cdots) + (a_1 + a_2 + \cdots)^2 + \cdots$$
$$= 1 + a_1 + (a_1^2 + a_2) + (a_1^3 + \underline{2a_1a_2}^{\,0} + a_3) + \cdots$$

In fact, the coeff on $a_1^{i_1} \cdots a_k^{i_k}$ in $\bar{a}$ is
$$\boxed{(i_1 + \cdots + i_k)! / i_1! \cdots i_k!}$$

(9) If $\xi \oplus \xi' =$ trivial then
$$w(\xi') = w(\xi)^{-1}$$

Note that (7) and (9) follow from (8).

<u>Ex</u>: Recall γ_n^1 is a subbundle of $\mathbb{R}P^n \times \mathbb{R}^{n+1} = \varepsilon^{n+1}$
$$E(\gamma_n^1) = \{([l], v) \mid l \text{ is a line in } \mathbb{R}^{n+1}, v \in l\}$$
Let $\gamma_n^\perp$ be the orthogonal complement of γ_n^1 in ε^{n+1}.
$$\gamma_n^\perp = \{([l], v) \mid v \text{ is perpendicular to } w \in l\}$$
0^*

Then $\gamma_n^1 \oplus \gamma_n^\perp = \varepsilon^{n+1} \Rightarrow$
$$w(\gamma_n^\perp) = \overline{w(\gamma_n^1)} = (1+a)^{-1} \quad \leftarrow \gamma_1^1 = i^*(\gamma_n^1)$$
$$= 1 + a + a^2 + \cdots + a^n \in H^*(\mathbb{R}P^n; \mathbb{Z}_2)$$
where $a \in H^1(\mathbb{R}P^n; \mathbb{Z}_2)$ is the generator.

Hence <u>all</u> of the Stiefel-Whitney classes of an $\mathbb{R}^n$-bundle may be non-zero.

Def: When M is a manifold, define $w_i(M) := w_i(TM)$

Ex: $S^n \subseteq \mathbb{R}^{n+1}$ hence $TS^n \oplus N(S^n \subseteq \mathbb{R}^{n+1}) = T\mathbb{R}^{n+1} \cong \varepsilon^{n+1}$

$$\Rightarrow w(S^n)\, w(\varepsilon^1) = w(\varepsilon^{n+1})$$

$$\Rightarrow w(S^n) = 1$$

Hence TS^n cannot be distinguished from trivial bundle using Stiefel-Whitney class (know $TS^2 \neq$ trivial)

Proof of (7): Let $f: B \longrightarrow G_n(\mathbb{R}^\infty)$ be a classifying map for ξ ($\xi = f^*(\gamma^n)$) so

$$w_{i,n}(\xi) = f^*(w_{i,n}), \quad w_{i,n} \in H^i(\mathbb{R}P^n; \mathbb{Z}_2)$$

We have a classifying map for $\xi \oplus \varepsilon$:

$$\begin{tikzcd}
 & \gamma^n \oplus \varepsilon \arrow[r] \arrow[d] & \gamma^{n+1} \arrow[d, "\pi"] \\
B \arrow[r, "f"] & G_n(\mathbb{R}^\infty) \arrow[r, "i_n"] & G_{n+1}(\mathbb{R}^\infty)
\end{tikzcd}$$

$$\xi \oplus \varepsilon \cong (i_n \circ f)^*(\gamma^{n+1})$$

$$\begin{aligned}
w_{i,n+1}(\xi \oplus \varepsilon) = (i_n \circ f)^*(w_{i,n+1}) &= f^*(i_n^*(w_{i,n+1})) \\
&= f^*(w_{i,n}) \\
&= w_{i,n}(\xi)
\end{aligned}$$