Day 21 - Cell structure for Grassmannian

Recall the subspaces

$$\mathbb{R}^0 \subseteq \mathbb{R}^1 \subseteq \mathbb{R}^2 \subseteq \cdots \subseteq \mathbb{R}^{m-1} \subseteq \mathbb{R}^m$$

where $\mathbb{R}^k$ has its last $m-k$ coordinates zero.

Given an n-plane $X \subseteq \mathbb{R}^m$, consider the sequence of integers:

$$0 \le \dim(X \cap \mathbb{R}^1) \le \dim(X \cap \mathbb{R}^2) \le \cdots \le \dim(X \cap \mathbb{R}^m) = n.$$

Say $X \cap \mathbb{R}^k$ is an l-dim subspace then either $X \cap \mathbb{R}^{k+1} = X \cap \mathbb{R}^k$ or has dimension one more than $X \cap \mathbb{R}^k$.

In fact

$$0 \to X \cap \mathbb{R}^k \longrightarrow X \cap \mathbb{R}^{k+1} \xrightarrow[k^{th}\text{ coord.}]{f_k} \mathbb{R} \longrightarrow 0$$

$$\Rightarrow \dim(X \cap \mathbb{R}^{k+1}) - \dim(X \cap \mathbb{R}^k) = \underset{\substack{\uparrow \\ \text{"jump at step } k\text{"}}}{\operatorname{rank}(f_k)} \in \{0,1\}.$$

<u>Def</u>: A Schubert symbol $\sigma = (\sigma_1, \dots, \sigma_n)$ is a sequence of n integers with

$$1 \le \sigma_1 < \sigma_2 < \cdots < \sigma_n \le m.$$

Let $e(\sigma) \subset G_n(\mathbb{R}^\infty)$ be the set of n planes X s.t.

$$\dim(X \cap \mathbb{R}^{\sigma_i}) = i, \ \dim(X \cap \mathbb{R}^{\sigma_i - 1}) = i-1$$

for all $1 \le i \le n$.

Since each $X \in G_n(\mathbb{R}^\infty)$ lives in precisely one $e(\sigma)$, $G_n(\mathbb{R}^\infty)$ is partitioned into

$$\{e(\sigma) \mid \text{all schubert symbols } \sigma\}.$$

Claim: $e(\sigma)$ is an open cell of dimension

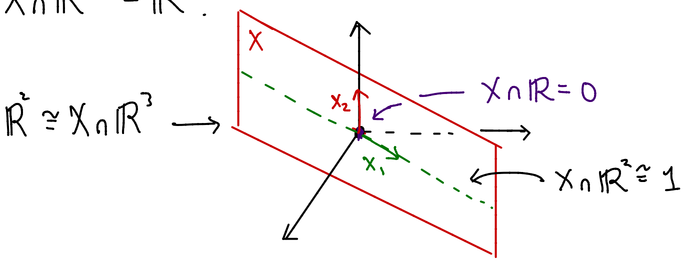
$$d(\sigma) = (\sigma_1 - 1) + \cdots + (\sigma_n - n).$$

Let $H^k \subseteq \mathbb{R}^k$ denote the open half space consisting of all $(\xi_1, \dots, \xi_k, 0, \dots)$ with $\xi_k > 0$.

Lemma: $X \in e(\sigma) \iff X$ has a basis $x_1 \in H^{\sigma_1}, \dots, x_n \in H^{\sigma_n}$

Proof: If X has such a basis then $\operatorname{rank}(f_{\sigma_k}) = 1$ $\Rightarrow X \in e(\sigma)$.

Suppose $X \in e(\sigma)$. We know $X \cap \mathbb{R} = \cdots = X \cap \mathbb{R}^{\sigma_1 - 1} = \{0\}$ and $X \cap \mathbb{R}^{\sigma_1} = \mathbb{R}$.



Pick $x_1 \in X \cap \mathbb{R}^{\sigma_1}$ as basis for $X \cap \mathbb{R}^{\sigma_1}$. Since $X \cap \mathbb{R}^{\sigma_1 - 1} = \{0\}$, the σ_1-coordinate is non-zero. Choose it to be pos. In fact, can choose it to be 1. Next consider $X \cap \mathbb{R}^{\sigma_1 + 1} = X \cap \mathbb{R}^{\sigma_1 + 2} = \cdots = X \cap \mathbb{R}^{\sigma_2 - 1}$ and $X \cap \mathbb{R}^{\sigma_2} \cong \mathbb{R}^2$.

Pick a basis x_1, x_2 of $X \cap \mathbb{R}^{\sigma_2}$ extending x_1. Since x_2 is lin. ind. of $\{x_1\}$, the σ_2-coordinate of x_2 is non-zero and hence we can assume it is positive. Continue in this way to complete the proof. $\boxtimes$

Note that $X \in e(\sigma) \iff X$ is the row space of a matrix of the form

$$
\begin{array}{cc}
n & \begin{pmatrix}
* & * & \cdots & * & 1 & 0 & \cdots & 0 & 0 & 0 \\
* & * & * & * & * & * & 1 & 0 & 0 & 0 \\
\vdots & & & & & & & & & \\
* & & & & & & * & * & 1 & 0
\end{pmatrix} \\
 & m
\end{array}
$$

1 is in the (i, σ_i) spot.

<u>Lemma</u>: Each $X \in e(\sigma)$ has unique orthonormal basis lying in $H^{\sigma_1} \times \cdots \times H^{\sigma_n}$.

<u>Proof</u>: Proceed like in the previous proof. Choose $x_1 \in \mathbb{R}^{\sigma_1} \cap X$ with unit length, there are two choices but only one with positive σ_1 entry. Now for x_2, choose x_2 to be orthogonal to x_1 and unique length. There are two but only one with positive σ_2-coordinate. Continue as before.

$\square$

<u>Def</u>: Let $e'(\sigma) = V_n^0(\mathbb{R}^m) \cap (H^{\sigma_1} \times \cdots \times H^{\sigma_n})$ be the set of orthormal frames with $x_i \in H^{\sigma_i}$. Let $\bar{e}'(\sigma)$ be the set w/ $x_i \in \bar{H}^{\sigma_i}$.

Lemma: $\bar{e}'(\sigma)$ is homeomorphic to a closed cell of dim $d(\sigma) = (\sigma_1 - 1) + \cdots + (\sigma_n - n)$ with interior $e'(\sigma)$, and the latter is mapped homeomorphically onto $e(\sigma)$ by the restriction $q: V_n^0(\mathbb{R}^m) \longrightarrow G_n(\mathbb{R}^m)$.

Idea of proof:

$\boxed{n=1}$ $\bar{e}'(\sigma_1) = \{(\xi_1, \ldots, \xi_k, 0, \ldots, 0) \mid \sum \xi_i^2 = 1,\ \xi_k \geq 0\}$,

closed upper hemisphere of $S^{\sigma_1 - 1} \Rightarrow$ homeo to $D^{\sigma_1 - 1}$!

Continue by induction (next time).