Day 22 - cell structure for $G_n(\mathbb{R}^\infty)$ con't.

Recall: $e'(\sigma) = V_n^0(\mathbb{R}^m) \cap (H^{\sigma_1} \times \cdots \times H^{\sigma_n})$ is the set of orthormal frames with $x_i \in H^{\sigma_i}$. Let $\bar{e}'(\sigma)$ be the set w/ $x_i \in \bar{H}^{\sigma_i}$.

(frame = ordered basis of $\mathbb{R}^m$)

Lemma: $\bar{e}'(\sigma)$ is homeomorphic to a closed cell of dim $d(\sigma) = (\sigma_1 - 1) + \cdots + (\sigma_n - n)$ with interior $e'(\sigma)$, and the latter is mapped homeomorphically onto $e(\sigma)$ via $q : \bar{e}'(\sigma) \longrightarrow G_n(\mathbb{R}^m)$.

Proof:

Prove by induction on n.

$\boxed{n=1}$ $\bar{e}'(\sigma_1) = \{(\xi_1, \ldots, \xi_k, 0, \ldots, 0) \mid \sum \xi_i^2 = 1,\ \xi_k \geq 0\}$, closed upper hemisphere of $S^{\sigma_1 - 1}$ $\Rightarrow$ homeo to $D^{\sigma_1 - 1}$!

Assume true for n. For any two vectors $u, v \in \mathbb{R}^m$ with $u \neq \pm v$, there is a unique rotation

$$T(u,v) : \mathbb{R}^m \longrightarrow \mathbb{R}^m$$

that sends $u \mapsto v$ and fixes everything orthogonal to the plane spanned by u, v. Specifically,

$$T(u,v)\,x = x - \frac{(u+v)\cdot x}{1 + u \cdot v}(u+v) + 2(u \cdot x)v.$$

T satisfies:

① $T(u,v)x$ is continuous as a function of 3 variables

② if $u,v \in \mathbb{R}^k$, then $T(u,v)x \equiv x \mod \mathbb{R}^k$.

Denote $b_i := (0,\dots,0,\underset{\substack{\uparrow \\ i\text{th}}}{1},0,\dots,0)$. Then $(b_1,\dots,b_n) \in e'(\sigma)$.

Let $(x_1,\dots,x_n)$ be any other n-frame in $e'(\sigma)$.

$$x_i = (\xi_1 \; \dots \; , \xi_i, 0 \cdots 0) \quad , \xi_i > 0$$

$$x_i \cdot x_j = \delta_{ij}$$

Consider the rotation of $\mathbb{R}^m$:

$$T = T(b_n, x_n) \circ T(b_{n-1}, x_{n-1}) \circ \cdots \circ T(b_1, x_1).$$

- $T(b_1,x_1), T(b_2,x_2), \dots, T(b_{i-1},x_{i-1})$ leave b_i fixed (since b_j, x_j has ith entry zero for $j<i$ so $x_j \cdot b_i = b_j \cdot b_i = 0$ for $j<i$).
- $T(b_i, x_i)$ sends b_i to x_i.
- $T(b_{i+1}, x_{i+1}), \dots, T(b_n, x_n)$ fix x_i (since $x_j \cdot x_i = 0$ for $i \neq j$ and jth entry of x_i is zero for $j > i$ so $x_i \cdot b_j = 0$)

Consider the image of b_i under T:

$$T(b_{i-1}, x_{i-1}) \circ \cdots \circ T(b_1, x_1) : b_i \longmapsto b_i$$

$$T(b_i, x_i) : b_i \longmapsto x_i$$

$$T(b_n, x_n) \circ \cdots \circ T(b_{i+1}, x_{i+1}) : x_i \longmapsto x_i.$$

Hence $T: b_i \longmapsto x_i$.

Let σ_{n+1} be an integer with $\sigma_{n+1} > \sigma_n$ and D be the set of unit vectors $u \in \bar{H}^{\sigma_{n+1}}$ with

$$b_1 \cdot u = \cdots = b_n \cdot u = 0$$

i.e. u is of form $(\overset{1}{0}, \dots, \overset{n}{0}, \overset{n+1}{*}, \dots, \overset{\sigma_{n+1}}{r}, 0, \dots, 0)$ with $r \geq 0$ in the σ_{n+1} coordinate and $u \cdot u = 1$.

The D is a closed hemisphere of dim $\sigma_{n+1} - n - 1$, and so is a closed ball. Define a homeomorphism

$$f : \bar{e}'(\sigma_1, \dots, \sigma_n) \times D \longrightarrow \bar{e}'(\sigma_1, \dots, \sigma_{n+1}) \text{ by}$$

$f((x_1, \dots, x_n), u) = (x_1, \dots, x_n, Tu)$, where T is defined using $(x_1, \dots, x_n)$ as above. By induction $\bar{e}'(\sigma_1, \dots, \sigma_n)$ is a closed ball of dimension $(\sigma_1 - 1) + \cdots + (\sigma_n - n)$. Hence $\bar{e}'(\sigma_1, \dots, \sigma_{n+1})$ is homeo to a closed ball of dimension $(\sigma_1 - 1) + \cdots + (\sigma_n - n) + (\sigma_{n+1} - (n+1))$.

First must show $\text{im}(f) \in \bar{e}'(\sigma_1, \dots, \sigma_{n+1})$.

Note: T is the rotation sending w/ $T(b_i) = x_i$ for $i \leq n$.

(1) $T(u)$ is a unit and orthogonal to x_i for $1 \leq i \leq n$.

To see this, note that T is a rotation hence preserves orthogonality. Thus since $b_i \cdot u = 0$ for $1 \leq i \leq n$ and $u \cdot u = 1$, we have $T(u) \cdot T(u) = u \cdot u = 1$ and $T(b_i) \cdot T(u) = b_i \cdot u = 0 \Rightarrow x_i \cdot T(u) = T(b_i) \cdot T(u) = 0$.

Hence $(x_1, \dots, x_n, Tu)$ is an n-frame in $\mathbb{R}^m$.

(2) $Tu \in \overline{H}^{\sigma_{n+1}}$.

To see this, recall that $b_i, x_i \in \mathbb{R}^{\sigma_n}$ for $i \le n$, so $Tu \equiv u \mod \mathbb{R}^{\sigma_n}$. Thus the σ_{n+1} coordinate of Tu is the same as the one for u, hence is non-negative.

f is clearly continuous. Moreover, T has an inverse

$$T^{-1} = T(x_1, b_1) \circ \cdots \circ T(x_n, b_n)$$

so f has a well-defined and cont. inverse.

We will omit the proof that

$$g|_{e(\sigma)} : e'(\sigma) \longrightarrow e(\sigma)$$

is a homeomorphism (see Milnor-Stasheff). We already showed that this is 1-1 and onto. (each n-plane $X \in e(\sigma)$ has a unique orthonormal basis in $H^{\sigma_1} \times \dots \times H^{\sigma_n}$). Recall g is the map which sends an n-frame in $\mathbb{R}^m$ to the span of the basis. $\boxtimes$

<u>Theorem</u>: The $\binom{m}{n}$ sets $e(\sigma)$ form the cells of a CW-complex with underlying space $G_n(\mathbb{R}^m)$. Taking the direct limit yields a CW-structure for $G_n(\mathbb{R}^\infty)$.

Need to show that each point in $\partial e(\sigma)$ belongs to a cell of lower dimension (read in book).

How many cells in each dimension?

Def: A partition of $r \in \mathbb{N}$ is an unordered sequence $i_1, \ldots, i_s$ with $i_1 + \cdots + i_s = r$. $p(r)$ is the number of partitions.

To each Schubert symbol $(\sigma_1, \ldots, \sigma_n)$ with

$$d(\sigma) = (\sigma_1 - 1) + \cdots + (\sigma_n - n) = r,$$

we have a partition of r (may need to cancel zeros at beginning of sequence), with $\sigma_i - i \le m-n$.

Cor: The number of r-cells in $G_n(\mathbb{R}^m)$ is equal the number of partitions of r where each integer is $\le m-n$.

- If $n, m-n \ge r$, this is $p(r)$.