<u>Day 23</u> - <u>Cohomology ring of $G_n(\mathbb{R}^\infty)$</u>

To prove this we shall assume the four axioms:

(1) To each rank n vector bundle $\xi : E \to B$, over a paracompact space B, $\exists$ $w_i(\xi) \in H^i(B;\mathbb{Z}_2)$ $\forall i$ with $w_0(\xi) = 1$ and $w_i(\xi) = 0$ for $i \geq n+1$.

(2) [Naturality] If $f : B' \to B$ is covered by a bundle map $\xi' \to \xi$ then $w_i(\xi') = f^* w_i(\xi)$.

(3) [Whitney sum] $w_i(\xi \oplus \xi') = w(\xi) w(\xi')$

(4) $w_1(\gamma_1^1) \neq 0$.

We will show then $H^*(G_n;\mathbb{Z}_n) \cong \mathbb{Z}_2[w_1,\dots,w_n]$ where $w_i = w_i(\gamma^n)$. For any $\xi : E \to B$, a rank n bundle, choose $f : B \to G_n$. By naturality,

$$w_i(\xi) = f^*(w_i(\gamma^n)) = f^*(w_i) \quad \text{(as before)}.$$

<u>Thm</u>: $H^*(G_n;\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1,\dots,w_n]$ with $w_i \in H^i(G_n;\mathbb{Z}_2)$

We first show that the ring generated by the w_i is free.

<u>Lemma</u>: There are polynomial relations among the w_i.

<u>Pf</u>: Suppose $p(w_1,\dots,w_n) = 0$ for some polynomial. Let $\xi : E \to B$ be any rank n-bundle then $\exists$ classifying map $f : B \to G_n$ s.t. $f^*(\gamma^n) \cong \xi$. By naturality,

$$w_i(\xi) = f^* w_i \quad \text{so}$$

$$p(w_1(\xi),\dots,w_n(\xi)) = f^* p(w_1,\dots,w_n) = 0.$$

We just need to find a bundle where there are no relations among $w_1(\xi), \dots, w_n(\xi)$. We know that $\mathbb{R}P^\infty = G_1(\mathbb{R}^\infty)$ and so

$$H^*(G_1(\mathbb{R}^\infty); \mathbb{Z}_2] = \mathbb{Z}_2[a]$$

with $a \in H^1(G_1; \mathbb{Z}_2)$. By Axiom 4, $w_1(\gamma_1^1) \neq 0$ so $w_1(\gamma_1^1) = a$.

Recall the cross product operation on cohomology with F=field,

$$H^p(X;F) \otimes H^q(Y;F) \xrightarrow{\times} H^{p+q}(X \times Y; F).$$
$$(a, b) \longmapsto p_1^*(a) p^*(b)$$

where $p_1 : X \times Y \to X$, $p_2 : X \times Y \to Y$. By Kunneth theorem for cohomology rings (p. 364 of Munkres), we have an isomorphism of algebras:

$$H^*(G_1; \mathbb{Z}_2) \otimes H^*(G; \mathbb{Z}_2) \xrightarrow{\times} H^*(G_1 \times G_1; \mathbb{Z}_2)$$

(need $H^i(G_1)$ f.g.). Hence

$$H^*(G_1 \times \cdots \times G_1; \mathbb{Z}_2) \cong H^*(G_1; \mathbb{Z}_2) \otimes \cdots \otimes H^*(G_1; \mathbb{Z}_2)$$

tensor algebra

$(a \otimes b) \cdot (c \otimes d) = ac \otimes bd$

Let $a_i = \pi_i(a)$ for π_i = proj. onto i^{th} factor and $a \neq 0$ in $H^1(G_1; \mathbb{Z}_2)$. Then $H^*(G_1 \times \cdots \times G_n)$ is a polynomial algebra on $a_1, \dots, a_n$. Thus all elements in $H^r(G_1 \times \cdots \times G_1; \mathbb{Z}_2)$ are sums of monomials of form $a_1^{s_1} \cdots a_n^{s_n}$ with $s_1 + \cdots + s_n = r$.

Let $\xi = \pi_1^*(\gamma^1) \oplus \cdots \oplus \pi_n^*(\gamma_1)$, then since $\pi_i : G_1 \times \cdots \times G_n \to G_1$, ξ is a bundle over $G_1 \times \cdots \times G_1$ (it is the product,

$\xi \cong \gamma' \times \cdots \times \gamma'$). Moreover

$$\begin{aligned} w(\xi) &= w(\pi_1^*(\gamma')) \cdots w(\pi_n^*(\gamma')) \qquad \text{[Axiom 3]} \\ &= \pi_1^*(w(\gamma')) \cdots \pi_n^*(w(\gamma')) \\ &= (1+a_1) \cdots (1+a_n) \end{aligned}$$

$$\Rightarrow w_1(\xi) = a_1 + a_2 + \cdots + a_n$$

$$w_2(\xi) = a_1a_2 + a_1a_3 + \cdots + a_1a_n + a_2a_3 + \cdots + a_{n-1}a_n.$$

$$w_n(\xi) = a_1 \cdots a_n.$$

In general, $w_i(\xi) =$ i^{th} elementary symmetric function of $a_1, \dots, a_n$. From algebra, these are known not to satisfy any relations. Thus neither do $w_1, \dots, w_n$.