Day 24: Cohomology of Grassmann, part II

We are trying to show that

$$H^*(G_n;\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1,\dots,w_n]$$

where $w_i = w_i(\gamma^n)$ satisfies the axioms.

Last time we showed that w_i satisfy no polynomial relations hence $\mathbb{Z}_2[w_1,\dots,w_n]$ is a subring of $H^*(G_n;\mathbb{Z}_2)$. We will show that they are the entire ring using a counting argument.

Recall: For a CW complex (f.g.) X

$$\#\text{ of } r\text{-cells of } X \geq \text{rank of } H^r(X;F) \text{ as a vector space}$$

From last time:

$$\#\text{ of } r\text{-cells of } G_n(\mathbb{R}^\infty) = \#\text{ of partitions of } r \text{ into at most } n \text{ integers.}$$
$(m=\infty)$.

Know $\operatorname{rank} H^r(G_n;\mathbb{Z}_2) \geq \#$ of monomials of dim r in $w_1,\dots,w_n$.

Such a monomial looks like:

$$w_1^{r_1}\cdots w_n^{r_n} \quad \text{where } r_1 + 2r_2 + \cdots + nr_n = r.$$

Claim: # of these monomials = # of partitions of n with $\leq n$ elements.

Proof of claim: Given $r_1, \dots, r_n$ with $r_1 + 2r_1 + \cdots + nr_n = r$ we have

$$\left.\begin{array}{l} r_1 + \\ r_2 + r_2 \\ r_3 + r_3 + r_3 \\ \quad \vdots \\ r_n + \cdots \qquad\quad + r_n \end{array}\right. = r$$

$$\Rightarrow (r_1 + \cdots + r_n) + (r_2 + \cdots + r_n) + \cdots + (r_{n-1} + r_n) + r_n = r$$

This gives a partition of r into at most n elements. (may need to delete zeros). Given a partition $s_1 + \cdots + s_n = r$ (some s_i may be zero). Can assume $s_1 \geq \cdots \geq s_n$. Let $r_n = s_n$, $r_{n-1} = s_{n-1} - s_n$, $r_{n-2} = s_{n-2} - s_{n-1}$, $\dots$, $r_2 = s_2 - s_3$, $r_1 = s_1 - s_2$. Then

$$\begin{aligned} & r_1 + 2r_2 + 3r_3 + \cdots + nr_n = \\ & \quad (s_1 - s_2) \\ & + 2(s_2 - s_3) \\ & + 3(s_3 - s_4) \\ & \quad \vdots \\ & + (n-1)(s_{n-1} - s_n) \\ & + n s_n \\ & = s_1 + s_2 + \cdots + s_n = r. \end{aligned}$$

∎

Consider the classifying map $g: G_1 \times \cdots \times G_1 \to G_n$.
for $\gamma^1 \times \cdots \times \gamma^1$. We have shown that

$$g^*: H^*(G_n; \mathbb{Z}_2) \longrightarrow H^*(G_1 \times \cdots \times G_1; \mathbb{Z}_2)$$
$$w_i \longmapsto s_i(a_1, \ldots, a_n) = \text{ith symm.}$$

maps $H^*(G_n; \mathbb{Z}_2)$ isomorphically onto the subalgebra of $H^*(G_1 \times \cdots \times G_1; \mathbb{Z}_2) \cong \mathbb{Z}_2[a_1, \ldots, a_n]$ consisting of polynomials in the indeterminants which are invariant under all permutations of S_n.

e.g. $\underbrace{(a_1a_2 + a_2a_3 + a_1a_3)}_{w_2}\underbrace{(a_1a_2a_3)}_{w_3}$ etc.

$\boxed{\text{Uniqueness of SW classes}}$

<u>Thm</u>: There exists at most one correspondence satifying the axioms (1)-(4), over paracompact spaces.

<u>Pf</u>: Say we have $\xi \to w(\xi)$ and $\xi \to \tilde{w}(\xi)$. For γ^1,

$$w(\gamma_1^1) = \tilde{w}(\gamma_1^1) = 1 + a.$$

Consider

$$\begin{tikzcd} i^*(\gamma^1) = \gamma_1^1 \arrow[r] \arrow[d] & \gamma^1 \arrow[d] \\ S^1 \arrow[r, hook, "i"] & G_1 \end{tikzcd} \Rightarrow$$

$$i^* w(\gamma^1) = w(i^*(\gamma^1)) = w(\gamma_1^1) = 1 + a$$

but since $i^*: H^1(G_1) \longrightarrow H^1(S^1)$ is an $\cong$ we have $w(\gamma^1) = 1 + a$. Similarly $\tilde{w}(\gamma^1) = 1 + a$.

Here $w_i(\gamma^1) = \tilde{w}_i(\gamma^1) = 0$ for $i > 1$ since it is a rank 1 vector bundle. For $\xi = \gamma^1 \times \cdots \times \gamma^1$, we have $w(\xi) = \tilde{w}(\xi) = (1+a_1)\cdots(1+a_n)$. Moreover, we have a bundle map $\xi \longrightarrow \gamma^n$ which maps $H^*(G_n;\mathbb{Z}_2) \hookrightarrow H^*(G_1 \times \cdots \times G_1;\mathbb{Z}_2) \Rightarrow w(\gamma^n) = \tilde{w}(\gamma^n)$.

For any n-plane bundle η over a paracompact space B, choose a classifying map $B \xrightarrow{f} G_n$. Then

$$w(\eta) = f^*(w(\gamma^n)) = f^*(\tilde{w}(\gamma^n)) = \tilde{w}(\eta)$$

$\blacksquare$

Existence Of SW Classes

We will construct SW classes using some operations called Steenrod square and the Thom isomorphism theorem. All homology/cohomology will be with $\mathbb{Z}_2$ coefficients.

Let $\xi : E \xrightarrow{\pi} B$ be a rank n vector bundle over B a closed n-dimensional vector bundle. [B just needs to be paracompact but we'll assume closed to simplify arguments.] Let $F = \pi^{-1}(b)$ be a fiber, $F_0 = F \setminus \{(b,\vec{0})\}$. From algebraic topology we have

$$H^i(F,F_0) \cong H^i(\mathbb{R}^n,\mathbb{R}^n) \cong H^i(B^n,\partial B^n) \cong H_{n-i}(B^n) \cong \begin{cases} \mathbb{Z}_2 & i=n \\ 0 & i \neq n. \end{cases}$$

$$\Rightarrow H^i(F,F_0;\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & i=n \\ 0 & i\neq n \end{cases}$$

Let $Z = \{(b,v) \mid b \in B,\ v \in \pi^{-1}(b)\}$ be the zero set and $E_0 = E \setminus Z$. We wish to compute $H^i(E,E_0)$. Put a Riemannian metric on B and set

$$\tilde{E} = \{(b,v) \mid b\in B,\ v\cdot v \leq 1\}$$

$$E_1 = \{(b,v) \mid b \in B,\ v\cdot v = 1\}.$$

Note that $\tilde{E}$ is a compact $(n+m)$-dimensional manifold that deformation retracts to B and $\partial\tilde{E} = E_1$. Hence

$$\begin{aligned} H^i(E,E_0) &\cong H^i(\tilde{E},\partial\tilde{E}) \\ &\cong H_{m+n-i}(\tilde{E}) \\ &\cong H_{m-(i-n)}(B) \\ &\cong H^{i-n}(B) \end{aligned}$$

<u>Note</u>: When $i<n$, $H^{i-n}(B)=0$ and when $n=i$, $H^{i-n}(B) = H^0(B) \cong \mathbb{Z}_2$.